Spatial coherence and Young-Michelson interferometry

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Abstract: The intensity distribution at the exit of a Young-Michelson interferometer is analysed in Fourier domain. It shows that two numbers are necessary for describing properly the variation of the visibility of the interferogram fringes. One of them is the complex degree of spatial coherence, which describes the correlation between the contributions from the Young’s slits. The second number describes the correlation between the Young’s interferograms reflected by the mirrors of the Michelson interferometer, that is, the correlation of optical fields that contain another correlation term.

Key words: Spatial coherence – Zernike’s formula – correlation function – Fourier transforms

1. Introduction

It has long been recognised that the term coherence plays a fundamental role in optics, where it is used at the present to denote the correlation properties to different orders of the optical field. Furthermore, it is accepted today that complete coherence requires significant correlation values to an infinite succession of orders [1, 2]. So, spatial coherence properties revealed by a simple Young’s experiment are referred as second order spatial coherence [2]. It describes the tendency of two values of the optical field at distantly separated points to take on correlated value [1]. Its basic quantity is the complex degree of spatial coherence [3].

Observation of higher order coherence properties, which are described by correlations of higher order, usually involves more complicated experimental situations. By example, 1955 Hanbury Brown and Twiss have performed a photon correlation experiment described by a quartic correlation [1, 2]. In such an experiment, the correlation between the photocurrents produced by intensity patterns collected by two different detectors is established.

On the other hand, as discussed in this paper, a cascade of a Young slit pair and a Michelson interferometer can provide more information about the structure of the second order spatial coherence of the optical field. Indeed, the fringe visibility of the intensity distribution of interferograms collected at the exit of this device is analysed, and it is shown that two numbers are necessary for describing properly the visibility of the fringe patterns.

One of them is the complex degree of spatial coherence, which describes the correlation between the contributions from the Young’s slits. The second number describes the correlation between the Young’s interferograms reflected by the mirrors of the Michelson interferometer, that is, the correlation between optical fields that contain a further correlation term. In this sense, the second number gives us more information about the spatial coherence properties of second order.

2. Basic theory

Fig. 1 depicts the experimental set-up we have used. A cascade of a Young slit pair and a spatially compensated Michelson interferometer (i.e. the optical path length of the both arms will differ only in less than the coherence length of the optical field) is disposed. In this way, a far field Young interferogram is reflected at the both mirrors of the Michelson interferometer.
The reflected interferograms will superimpose at a CCD sensor, located at the exit of the whole device, which records the corresponding intensity distribution.

Tilting one of the mirrors of the Michelson interferometer, say M2, can perform relative displacements between the superimposed interferograms. It allows us to observe the spatial coherence properties of the optical field, in which we are concerning.

The intensity distribution recorded by the CCD sensor should be obtained from the autocorrelation of the amplitude distribution of the optical field there [3], that is $I(x, y) = W(x, y)$, where

$$W(x_1, y_1; x_2, y_2) = \langle V(x_1, y_1), V^*(x_2, y_2) \rangle,$$

and $\langle \rangle$ symbolizes the correlation operation. It is clear $V(x, y) = A_1(x, y) + A_2(x, y)$, with $A_j(x, y)$ ($j = 1, 2$) the amplitude of the contribution reflected by the $j$-th mirror of the Michelson interferometer, so that

$$W(x_1, y_1; x_2, y_2) = \langle A_1(x_1, y_1), A_1^*(x_2, y_2) \rangle + \langle A_2(x_1, y_1), A_2^*(x_2, y_2) \rangle + 2 \Re \{\langle A_1(x_1, y_1), A_2^*(x_2, y_2) \rangle\}$$

where asterisk denotes complex conjugate and $\Re$ the real part. The correlation $(A_j(x_1, y_1), A_j^*(x_2, y_2))$ can be determined by using the Zernike's formula in Fraunhofer domain (far field approach) [3]. Specifically,

$$\langle A_j(x_1, y_1), A_j^*(x_2, y_2) \rangle = \left( \frac{1}{2\pi} \right)^2 I_0 \int \int \mu_0(\xi_1 - \xi_2; \eta_1 - \eta_2) t(\xi_1, \eta_1) r^*(\xi_2, \eta_2)$$

$$- e^{-i \frac{\pi}{\lambda} (x_1 + x_2; y_1 + y_2)} \right) d\xi_1 d\eta_1 d\xi_2 d\eta_2.$$

Under the above conditions, eq. (3) yields straightforwardly

$$\langle A_1(x_1, y_1), A_1^*(x_2, y_2) \rangle = 2 \left( \frac{ab}{\lambda z} \right)^2 I_0 \sin \left( \frac{ka}{2z} x_1 \right)$$

$$\cdot \sin \left( \frac{kb}{2z} x_2 \right) \sin \left( \frac{ka}{2z} y_1 \right) \sin \left( \frac{kb}{2z} y_2 \right)$$

with $\sin(ax) = \frac{\sin(ax)}{ax}$ [4]. A similar formula is obtained for $\langle A_2(x_1, y_1), A_2^*(x_2, y_2) \rangle$, but in this case a phase factor should be introduced into the integrand of the Zernike's formula too describe properly the relative displacement of the interferograms at the CCD sensor. Therefore,

$$\langle A_2(x_1, y_1), A_2^*(x_2, y_2) \rangle = 2 \left( \frac{ab}{\lambda z} \right)^2 I_0$$

$$\cdot \sin \left( \frac{ka}{2z} (x_1 - X) \right) \sin \left( \frac{ka}{2z} (x_2 - X) \right)$$

$$\cdot \sin \left( \frac{kb}{2z} (y_1 - Y) \right) \sin \left( \frac{kb}{2z} (y_2 - Y) \right)$$

$$\cdot \left\{ \cos \left( \frac{kc}{2z} (x_1 - x_2) \right) + \mu \cos \left( \frac{kc}{2z} (x_1 + x_2 - 2X) \right) \right\}$$

where $(X, Y)$ denotes the relative displacement between the interferograms at the CCD sensor plane.

On the other hand, it is useful to introduce the normalised cross-correlation

$$\nu(x_1, y_1; x_2, y_2) = \frac{\langle A_1(x_1, y_1), A_1^*(x_2, y_2) \rangle}{\sqrt{I_1(x_1, y_1)} \sqrt{I_2(x_2, y_2)}}.$$

The function $I_j(x_1, y_1) = \langle A_j(x_1, y_1), A_j^*(x_2, y_2) \rangle$ ($j = 1, 2$) is the intensity distribution of the Young interferogram reflected by the $j$-th mirror at the CCD sensor. These intensity distributions can be expressed as

$$I_1(x_1, y_1) = 2 \left( \frac{ab}{\lambda z} \right)^2 I_0 \sin^2 \left( \frac{ka}{2z} x_1 \right) \sin^2 \left( \frac{kb}{2z} y_1 \right)$$

$$\cdot \left\{ 1 + \mu |\cos \left( \frac{kc}{2z} x_1 \right) \right\}$$

and

$$I_2(x_2, y_2) = 2 \left( \frac{ab}{\lambda z} \right)^2 I_0 \sin^2 \left( \frac{ka}{2z} x_2 - X \right)$$

$$\cdot \sin^2 \left( \frac{kb}{2z} y_2 - Y \right) \left\{ 1 + \mu |\cos \left( \frac{kc}{2z} x_2 - X \right) \right\}.$$

The normalised cross-correlation (6) describes the correlation degree between the contributions reflected by the mirrors of the Michelson interferometer. Indeed, in far field approach it takes the form

$$\nu(x_1, y_1; x_2, y_2) = \left| \nu(x_1, y_1; x_2, y_2) \right|$$

$$- e^{-i \frac{k}{2z} \{ (x_1 + x_2) \cdot (y_1 + y_2) \} + \alpha}.$$
whose modulus $0 \leq |v(x_1, y_1; x_2, y_2)| \leq 1$ and phase $\alpha_{12}$ will depend on the separation between the points $(x_1, y_1)$ and $(x_2, y_2)$, so that $\alpha_{ij} = 0$.

It is reasonable to assume that $|v|$ and $\alpha_{12}$ are constant for a fixed slit pair and for a specific position of the tilted mirror in our experiment. As before, we assume $\alpha_{12} = 0$ for the sake of simplicity and without lack of generality. Under this condition and taking into account eq. (6), the last term in eq. (3) in far field approach can be written as

$$ 2 \Re \left[ A_i(x_1, y_1) \cdot A^*_j(x_2, y_2) \right] = 2 \sqrt{I_1(x_1, y_1)} \cdot \sqrt{I_2(x_2, y_2)} \Re \left[ v(x_1, y_1; x_2, y_2) \right] $$

Furthermore, the intensity distribution recorded by the CCD sensor can be obtained from eqs. (2), (4), (5) and (8) as

$$ I(x, y) = W(x, y; x, y) = 2 \left( \frac{ab}{z} \right)^2 I_0 \left[ \sin^2 \left( \frac{ka}{2z} y \right) \sin^2 \left( \frac{kb}{2z} \right) \right] $$

From eq. (9) we infer that $v(x_1, y_1; x_2, y_2)$ is a different quantity from the complex degree of spatial coherence $\mu_0(\xi_1 - \xi_2; \eta_1 - \eta_2)$, which describes the correlation between the contributions from the Young’s slits. So, $v(x_1, y_1; x_2, y_2)$ describes the correlation of optical fields that contain a correlation term due a prior Young’s interference. Furthermore, these two quantities are necessary to describe properly the intensity distribution of the interferograms.

### 3. Simulations, experimental results and discussion

Fig. 2 shows simulated intensity distributions of interferograms described by eq. (9). They were calculated for different values of $|v|$ with $|\mu| = 1$ and $X = Y$. The loss of visibility of the Michelson fringes with the decrease of $|v|$ is apparent whereas the high visibility of the Young fringes remains.

Fourier analysis of the interferograms was applied to perform the evaluation of $|\mu|$ and $|v|$ separately. The Fourier spectrum of their intensity distributions is given by

$$ \tilde{I}(\xi, \eta) = \int I(x, y) e^{i \frac{2\pi}{z} (\xi x + \eta y)} dx dy $$

$$ = \frac{1}{\pi^2} I_0 \left\{ \text{tri} \left( \frac{\xi}{a}, \frac{\eta}{b} \right) \right\} $$

$$ \otimes \left\{ \delta(\xi, \eta) + \frac{|\mu|}{2} \left[ \delta(\xi + c, \eta) + \delta(\xi - c, \eta) \right] \right\} $$

$$ + \text{tri} \left( \frac{\xi}{a}, \frac{\eta}{b} \right) e^{i \frac{2\pi}{z} (\xi x + \eta y)} $$

$$ \otimes \left\{ \delta(\xi, \eta) + \frac{|\mu|}{2} e^{i \frac{2\pi}{z} (\xi x + \eta y)} \right\} $$

$$ \otimes H(\xi) \delta(\eta) + H(\xi) \delta(\eta) e^{i \frac{2\pi}{z} (\xi x + \eta y)} $$

where $R$ symbolises convolution [4] and the functions that appear in eq. (10) are defined as follows [4]:

$$ \text{tri} \left( \frac{\xi}{a}, \frac{\eta}{b} \right) = \left\{ \begin{array}{ll} 1 - |\xi| & \text{for } a |\xi| \leq b \\
|\eta| & \text{otherwise} \\
0 & \text{otherwise} \end{array} \right. $$

with $R$ the region limited by the four straight lines $|\xi| = a$ and $|\eta| = b$. 

Fig. 2. Simulated interferograms at the exit of a Young-Michelson interferometer.
• \( \delta(\xi, \eta) \) is the Dirac’s delta function,

\[
\text{rect}\left(\frac{\xi}{a}, \frac{\eta}{b}\right) = \begin{cases} 
1 & |\xi| \leq \frac{a}{2} \text{ and } |\eta| \leq \frac{b}{2}, \\
0 & \text{otherwise}
\end{cases}
\]

• \( H(\xi) \) is the Fourier spectrum of \( 1+|\mu|\cos\left(\frac{k_c}{z}x\right) \).

It is useful to expand it in series before the calculation of its Fourier spectrum. Thus, we have [5]

\[
\sqrt{1+|\mu|\cos\left(\frac{k_c}{z}x\right)} = \sum_{n=0}^{\infty} \beta_n |\mu|^{2n} \cos\left(\frac{2n}{n}x\right)
\]

\[
= 1 + \sum_{n=1}^{\infty} \left( \frac{\beta_{2n}}{2^{2n}} |\mu|^{2n} \sum_{j=0}^{n-1} \left( \begin{array}{c} n \\ j \end{array} \right) \cos\left(\frac{2n}{n}x\right) + \beta_{2n-1} \frac{2^{2(n-1)}}{2^2} |\mu|^{2n-1} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \\ j \end{array} \right) \cos\left(\frac{(n-j-1)}{2}\frac{2n}{n}x\right) \right)
\]

where the coefficients \( \beta_n \) are constants: \( \beta_0 = 1 \), \( \beta_1 = \frac{1}{2} \), \( \beta_2 = \frac{1}{8} \), \( \beta_3 = \frac{3}{48} \), \( \beta_4 = -\frac{15}{384} \). Therefore, the Fourier spectrum of this expression is

\[
H(\xi) = \delta(\xi) + \sum_{n=1}^{\infty} \left( \frac{\beta_{2n}}{2^{2n}} |\mu|^{2n} \left( \begin{array}{c} 2n \\ n \end{array} \right) \delta(\xi)
\]

\[
+ \sum_{j=0}^{n-1} \left( \begin{array}{c} n \\ j \end{array} \right) [\delta(\xi + 2(n-j)c) + \delta(\xi - 2(n-j)c)]
\]

\[
+ \beta_{2n-1} \frac{2^{2(n-1)}}{2^2} |\mu|^{2n-1} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \\ j \end{array} \right) \delta(\xi + 2(n-j-1)c) + \delta(\xi - 2(n-j-1)c)]
\]

\[= \delta(\xi) + \sum_{n=1}^{\infty} \left( \frac{\beta_{2n}}{2^{2n}} |\mu|^{2n} \left( \begin{array}{c} 2n \\ n \end{array} \right) \delta(\xi)
\]

\[
+ \sum_{j=0}^{n-1} \left( \begin{array}{c} n \\ j \end{array} \right) [\delta(\xi + 2(n-j)c) + \delta(\xi - 2(n-j)c)]
\]

\[
+ \beta_{2n-1} \frac{2^{2(n-1)}}{2^2} |\mu|^{2n-1} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \\ j \end{array} \right) \delta(\xi + 2(n-j-1)c) + \delta(\xi - 2(n-j-1)c)]
\]

\[= \delta(\xi) + \sum_{n=1}^{\infty} \left( \frac{\beta_{2n}}{2^{2n}} |\mu|^{2n} \left( \begin{array}{c} 2n \\ n \end{array} \right) \delta(\xi)
\]

\[
+ \sum_{j=0}^{n-1} \left( \begin{array}{c} n \\ j \end{array} \right) [\delta(\xi + 2(n-j)c) + \delta(\xi - 2(n-j)c)]
\]

\[
+ \beta_{2n-1} \frac{2^{2(n-1)}}{2^2} |\mu|^{2n-1} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \\ j \end{array} \right) \delta(\xi + 2(n-j-1)c) + \delta(\xi - 2(n-j-1)c)]
\]

\[= \delta(\xi) + \sum_{n=1}^{\infty} \left( \frac{\beta_{2n}}{2^{2n}} |\mu|^{2n} \left( \begin{array}{c} 2n \\ n \end{array} \right) \delta(\xi)
\]

\[
+ \sum_{j=0}^{n-1} \left( \begin{array}{c} n \\ j \end{array} \right) [\delta(\xi + 2(n-j)c) + \delta(\xi - 2(n-j)c)]
\]

\[
+ \beta_{2n-1} \frac{2^{2(n-1)}}{2^2} |\mu|^{2n-1} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \\ j \end{array} \right) \delta(\xi + 2(n-j-1)c) + \delta(\xi - 2(n-j-1)c)]
\]

(12)

From eqs. (10) to (12) we conclude that \( \tilde{I}(\xi, \eta) \) will consists of a set of peaks, distributed as follows (fig. 3):

- The first two terms in eq. (10) are corresponding to the Young interferograms, without regarding the modulation by Michelson fringes. They consist of three peaks with pyramidal profiles, one of them at the origin of coordinates (\( \xi, \eta \)) and the other two symmetrically located on the \( \xi \)-axis at a distance \( c \) from the origin. Note that the positions of corresponding peaks in both terms coincide and that the lateral peak height is proportional to \( |\mu| \).

- The third term in eq. (10) provides the information about the modulation of the Young interferograms by Michelson fringes. This term describes two sets of peaks with pyramidal profiles, given by the convolution of the rect functions. These sets are located symmetrically with respect to the origin of co-ordinates at (\( X, Y \)) and (\( -X, -Y \)) respectively, and their peaks are distributed according to \( H(\xi) \).

- Each set has a principal peak, given by the first Dirac’s delta function of \( H(\xi) \). Its height provides separately information about \( |\mu| \). The remained peaks are distributed by pairs and symmetrically with respect to the principal one, but their heights decay rapidly because they are proportional to products of the form \( |\mu||\mu|^m \), with \( m \) an integer (fig. 3).

Therefore, for measuring purposes let us consider the principal peak of one of the sets, say that located at (\( X, Y \)). Its mathematical form is given by [eq. (10)]

\[
\frac{I_0}{\pi^2} \frac{|\mu|}{2} \text{rect}\left(\frac{\xi}{a}, \frac{\eta}{b}\right) \otimes \text{rect}\left(\frac{\xi}{a}, \frac{\eta}{b}\right) e^{i\frac{\xi}{2}X + i\frac{\eta}{2}Y}
\]

\[
\otimes \delta(\xi, \eta) \otimes \delta(\xi, \eta) e^{i\frac{\xi}{2}X + i\frac{\eta}{2}Y} \otimes \delta(\xi - X, \eta - Y).
\]

It is apparent that the height of this peak is proportional to \( |\mu| \).

In summary, by looking at only two peaks of \( \tilde{I}(\xi, \eta) \) we can obtain separate measurements of \( |\mu| \) and \( |\mu| \).

Fig. 4 shows experimental interferograms obtained by illuminating two different Young’s slit pair with a He-Ne laser and a conventional laser diode respectively. The module of their Fourier spectra is shown in fig. 5 and the corresponding values for \( |\mu| \) and \( |\mu| \) are specified in table 1.

The high fringe visibility of the interferograms obtained by He-Ne laser in both situations is apparent. In the first one, the value of \( |\mu| \) is close to 1 and only diminishes in 12.4% when the separation of the Young’s slit pair increases. The value \( |\mu| \) is high but not close to 1 in the first situation, and diminishes in 13.7% in the second situation.

But the visibility of the Michelson fringes decreases significantly by laser diode. In the first situation, the value of \( |\mu| \) is close to 1 and diminishes in 17.6% when the separation of the Young’s slit pair increases. But the both values of \( |\mu| \) are small and the second is 27.5% smaller than the first.

The visibility of the Young fringes produced by the both light sources by the Young’s pair with the closest separa-
tion differs only in 4.2%, but that of the Michelson fringes shows a difference of 50%, which is apparent in fig. 4a, c. By the second Young’s pair the situation is relative similar. The visibility of the Young fringes differs in 10.3% whereas that of the Michelson fringes differs in 58.0%.

These results evidence the different meaning and separability of \( m_0 (x_1 - x_2; h_1 - h_2) \) and \( \nu (x_1, y_1; x_2, y_2) \) as descriptors of the physical properties of light sources. As a consequence, both quantities are necessary for describing properly the spatial coherence of the optical field, revealed by the interferograms. We also conclude that \(|\nu|\) describes the correlation between optical fields that contain a correlation term given by the prior Young’s interference.

Fig. 4. Experimental interferograms at the exit of a Young-Michelson Interferometer: Illuminations: a) and b) He-Ne laser, c) and d) Laser diode. Young’s slit pairs: a) and c) width 0.1 mm, separation 0.2 mm, b) and d) width 0.1 mm, separation 0.3 mm.

This work was performed at the Abdus Salam International Centre for Theoretical Physics (AS-ICTP). The authors express their acknowledgments to the AS-ICTP for the financial support and to Prof. Franco Gori for his inspiring discussions. Jorge García-Sucerquia thanks the Fundación Mazda para el Arte y la Cultura and the Fondo para apoyar Trabajos de Grado de Pre- y Posgrado de la Universidad de Antioquia. One of the authors, Francisco F. Medina-Estrada, undertook this work with the support of the “ICTP Programme for Training and Research in Italian Laboratories”.

Table 1. Experimental results for \(|\mu|\) and \(|\nu|\) by using two different light sources and Young’s slit pairs.

<table>
<thead>
<tr>
<th>Young’s Slit Pair</th>
<th>He-Ne Laser</th>
<th>Laser Diode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(</td>
<td>\mu</td>
</tr>
<tr>
<td>width 0.1 mm</td>
<td>0.949±0.006</td>
<td>0.800±0.005</td>
</tr>
<tr>
<td>separation 0.2 mm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>width 0.1 mm</td>
<td>0.831±0.006</td>
<td>0.690±0.005</td>
</tr>
<tr>
<td>separation 0.3 mm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. Fourier spectra of the experimental interferograms in fig. 4. Illuminations: a) and b) He-Ne laser, c) and d) Laser diode. Young’s slit pairs: a) and c) width 0.1 mm, separation 0.2 mm, b) and d) width 0.1 mm, separation 0.3 mm.
References


