Unified reconstruction theory for diffraction tomography, with consideration of noise control

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In diffraction tomography, the spatial distribution of the scattering object is reconstructed from the measured scattered data. For a scattering object that is illuminated with plane-wave radiation, under the condition of weak scattering one can invoke the Born (or the Rytov) approximation to linearize the equation for the scattered field (or the scattered phase) and derive a relationship between the scattered field (or the scattered phase) and the distribution of the scattering object. Reconstruction methods such as the Fourier domain interpolation methods and the filtered backpropagation method have been developed previously. However, the underlying relationship among and the noise properties of these methods are not evident. We introduce the concepts of ideal and modified sinograms. Analysis of the relationships between, and the noise properties of the two sinograms reveals infinite classes of methods for image reconstruction in diffraction tomography that include the previously proposed methods as special members. The methods in these classes are mathematically identical, but they respond to noise and numerical errors differently. © 1998 Optical Society of America

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1. INTRODUCTION

Diffraction tomography is a technique that reconstructs and reveals the spatially variant distribution of appropriate physical parameters (e.g., the distribution of the refractive index in ultrasound diffraction tomography) in an object from the measured scattered field.1-4 This technique has attracted much attention in the past two decades because of its great potential for wide application in different scientific disciplines, ranging from medical imaging to geophysics.5-11 For example, there has been increasing interest in the application of ultrasound diffraction tomography to breast tissue imaging.12,13

Unlike the x rays in computed tomography (CT), which travel in a straight line, the radiation in diffraction tomography has to be treated in terms of wave fronts and fields scattered by inhomogeneities in the object.14 Therefore a wide variety of techniques suitable for reconstruction of CT images cannot be used directly for reconstruction of diffraction tomographic images.15,16 In diffraction tomography the interaction between the radiation and the object medium is governed by the inhomogeneous Helmholtz equation.17,18 Solution of this inhomogeneous equation yields a relationship between the scattered field and the distribution of the physical parameters of the object (e.g., the refractive-index distribution in ultrasound diffraction tomography). Using this relationship, one can subsequently derive the closed-form relationship between the measured quantity (i.e., the scattered field in the Born approximation or the unwrapped complex phase between the measured quantity (i.e., the scattered field in the Rytov approximation) and the distribution of the physical parameters of the object.

The reconstruction of diffraction tomographic images has been described extensively in the literature. According to the generalized central-slice theorem (also called the Fourier diffraction projection theorem), when an object is illuminated with plane-wave radiation of monochromatic frequency, under the Born (or the Rytov) approximation the one-dimensional (1D) Fourier transform of the measured scattered field (or unwrapped phase) can be related to the two-dimensional (2D) Fourier transform of the object function.24 Reconstruction techniques have been devised that are based on this generalized central-slice theorem. Of particular importance are the Fourier-domain interpolation24 and the filtered backpropagation (FBPP) methods.25 Recently some studies were reported on the noise properties of the reconstruction methods in diffraction tomography.26-29

A medical imaging modality called single-photon-emission computed tomography (SPECT) collects γ rays
emitted from radioactive isotopes that are administered to the object. 30,31 From the collected projections of γ rays, the distribution of the isotopes in the object can be reconstructed. However, γ rays unavoidably undergo attenuation in the object, and hence only a portion of them can eventually reach the detector. Therefore projections acquired in SPECT are attenuated, collected over all angles, and constitute the attenuated Radon transform or attenuated sinogram. When the object has a uniform distribution of the attenuation coefficient in 2D SPECT, the attenuated Radon transform can be transformed into the so-called exponential Radon transform or the modified sinogram. 32,33 Reconstruction from the attenuated sinogram without adequate compensation for the effect of attenuation yields SPECT images that are visually distorted and quantitatively inaccurate. Therefore one of the challenging tasks in SPECT reconstruction is to estimate, from the attenuated sinogram, the ideal sinogram, which is free of any attenuation effects and can be used for reconstruction of distortion-free and quantitatively accurate SPECT images. For 2D SPECT with uniform attenuation, we have proposed a strategy to explore the implication of data noise for reconstruction, and we have developed a unified theory and an infinite class of methods for ideal-sinogram estimation and image reconstruction. 34,35

The mathematical form for the 1D Fourier transform of the modified sinogram in 2D SPECT is identical to that for the 2D Fourier transform of the objective function along a semicircle in its 2D Fourier space in 2D diffraction tomography. In this paper, therefore, we will apply the strategy and concepts that were developed for 2D SPECT with uniform attenuation to 2D diffraction tomography to the investigation of the implications of the data noise on the reconstructed object function and to the development of a unified theory for ideal-sinogram estimation and the FBPP methods.

This work is presented as follows. In Section 2 the Born and the Rytov approximations are briefly reviewed, the concepts of the ideal and the modified sinograms are introduced, and the connection between the modified sinogram and the measured quantity in 2D diffraction tomography is made. In Section 3 a 2D Fourier analysis of the ideal and the modified sinograms is conducted and the relationships between the 2D Fourier transforms of the ideal and the modified sinograms are established. These relationships may be used for estimation of the ideal sinogram from the modified sinogram, which, in turn, can be obtained from the measured data. In Section 4 the measured quantity is treated as a stochastic process, and an investigation is done of the properties of its second-order statistics and the implications of these properties for the estimation of the ideal sinogram as well as for the reconstructed object function. In Section 5, with the insights gained in Section 4, a linear combination approach is proposed for estimation of the 2D Fourier transform of the ideal sinogram, and an exploration is done of the optimal combination coefficient that yields a bias-free estimate with minimum variance, which leads to the discovery of an infinite class of ideal-sinogram estimation methods. Section 6 presents the development of an infinite class of generalized filtered backpropagation (GFBPP) methods, which includes the FBPP method proposed by Devaney 25 as a special member, and a counterpart relationship between the GFBPP methods and the ideal-sinogram estimation methods is established. Furthermore, it is demonstrated that the ideal-sinogram estimation method that corresponds to the FBPP method is generally statistically suboptimal. In Section 7 a discussion and conclusions are presented.

2. SINOGRAMS

In 2D tomographic imaging the measured projection data are used to form the so-called sinogram, from which the tomographic image of the object may be reconstructed.

A. Ideal Sinogram

The ideal sinogram, i.e., the Radon transform, 15,36 is formed from a set of projections (or line integrals) of an object function \( a(\vec{r}) \) along a set of parallel straight lines at different projection angles, as shown in Fig. 1(a). Mathematically, the ideal sinogram (or the Radon transform) of \( a(\vec{r}) \) is expressed as

\[
\mathcal{R}[a(\vec{r})] = \int_{-\infty}^{\infty} a(x, y) \delta(\vec{r} - \vec{r}(x, y)) d\vec{r}
\]

Fig. 1. (a) 2D Radon transform \( \mathcal{R}(\xi, \phi) \) is an integral of the 2D object function \( a(\vec{r}) \) along a straight line, which is specified by two parameters, \( \xi \) (the distance of the line to the origin of the coordinate systems) and \( \phi \) (the angle between the \( \xi \) and \( x \) axes). (b) According to the central-slice theorem, the 1D Fourier transform \( P(\nu_x, \phi) \) of the Radon transform equals the 2D Fourier transform \( A(\nu_x, \nu_y) \) of the object function along a straight line with a slope of \( \tan \phi \) in the 2D Fourier space.
\[ p(\xi, \phi_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\vec{r}) \delta[\xi - r \cos(\phi_0 - \theta)] d\vec{r}, \]

(1)

where \( \phi_0 \) is the projection angle, \( x = r \cos \theta, y = r \sin \theta, \xi = x \cos \phi_0 + y \sin \phi_0 = r \cos(\phi_0 - \theta) \), and \( \eta = -x \sin \phi_0 + y \cos \phi_0 = -r \sin(\phi_0 - \theta) \). In the absence of noise and the effect of finite sampling, the function \( a(\vec{r}) \) can be reconstructed exactly from the ideal sinogram \( p(\xi, \phi_0) \) by any of a wide variety of conventional reconstruction techniques such as the filtered backprojection (FBP) technique.\(^{15,36}\)

The ideal sinogram of Eq. (1) can be treated as a 2D function of \( \xi \) and \( \phi_0 \), and hence its 1D Fourier transform with respect to \( \xi \) is given by

\[ P(v_a, \phi_0) = \int_{-\infty}^{\infty} p(\xi, \phi_0) \exp(-j2\pi v_a \xi) d\xi \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\vec{r}) \exp[-j2\pi v_a r \cos(\phi_0 - \theta)] d\vec{r}, \]

(2)

where \( v_a \) is the spatial frequency of the ideal sinogram. According to the central-slice theorem, \( P(v_a, \phi_0) \) can be interpreted as the 2D Fourier transform of the object function \( a(\vec{r}) \) evaluated along a straight line with a slope of \( \tan \phi_0 \) in the 2D Fourier space of the object function, as shown in Fig. 1(b).

B. Modified Sinogram

In diffraction tomography, an object immersed in an absorberless and homogeneous background medium (such as water) is illuminated by plane-wave radiation of monochromatic frequency \( v_0 > 0 \) propagating along the \( \eta \) axis, as shown in Fig. 2(a).

1. Helmholtz Equation

In the absence of the object, the plane-wave radiation propagates through the absorberless and homogeneous background medium and has an incident field \( u_\eta(\xi, \phi) = U_0 \exp(-j2\pi v_0 \eta) \) at \( \eta = l \), where \( \xi = r \cos(\phi - \theta) \) and \( U_0 \) is the field of the plane wave at the center of rotation of the coordinate system.\(^{24}\) When an object with a spatially varying object function \( a(\vec{r}) \) is immersed in the background medium, scattering occurs with a scattered field \( u_s(\xi, \phi) \) at \( \eta = l \). Note that \( U_0, u_\eta(\xi, \phi), \) and \( u_s(\xi, \phi) \) are generally complex numbers. The measured scattered field \( u_s \) is related to the object function \( a(\vec{r}) \) by the inhomogeneous Helmholtz equation\(^{1,14}\)

\[ \nabla^2 u_s + 4\pi^2 v_0^2 u_s = -4\pi^2 v_0^2 a(\vec{r})(u_i + u_s), \]

(3)

where \( \nabla^2 \) is the Laplace operator that operates on the spatial variable \( \vec{r} \).

In ultrasound diffraction tomography, the refractive-index function \( n(\vec{r}) \) of the object is of interest. The object function \( a(\vec{r}) \) is related to \( n(\vec{r}) \) through \( a(\vec{r}) = n^2(\vec{r}) - 1 \) and may be complex. \( a(\vec{r}) \) is also called the differential acoustical refractive index. However, for an absorberless object, the refractive index function \( n(\vec{r}) \) is real, as is the object function \( a(\vec{r}) \). The task of reconstructing the index function \( n(\vec{r}) \) is equivalent to that of reconstructing the object function \( a(\vec{r}) \). The solution to the Helmholtz equation [Eq. (3)] provides the relationship between \( u_s \) and \( a(\vec{r}) \), which can be used subsequently to reconstruct \( a(\vec{r}) \) in terms of \( u_s \). Hence the first step is to solve the Helmholtz equation [Eq. (3)] for \( u_s \) in terms of \( a(\vec{r}) \).

2. Born and Rytov Approximations

It is virtually impossible to solve the inhomogeneous Helmholtz equation [Eq. (3)] exactly to obtain a relationship between \( u_s \) and an arbitrary object function \( a(\vec{r}) \). However, under the condition of weak scattering, appropriate approximations can be made in Eq. (3) so that analytic relationships between \( u_s \) and \( a(\vec{r}) \) can be derived.

**Born approximation.** Let \( U_s(v_m, \phi) = \int_{-\infty}^{\infty} u_s(\xi, \phi) \times \exp(-j2\pi v_m \xi) d\xi \) be the 1D Fourier transform of the scattered field \( u_s(\xi, \phi) \) with respect to \( \xi \). In the Born approximation the scattered field \( u_s \) on the right-hand side of Eq. (3) is ignored if we assume \( |u_s| \ll |u_i| \). It can be shown that solving Eq. (3) under the Born approximation yields a relationship between \( U_s(v_m, \phi) \) and the object function \( a(\vec{r}) \), which is given by\(^{1,14,24}\)

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**Fig. 2.** (a) In 2D diffraction tomography, one measures at \( \eta = l \) the scattered field from a scattering object that is illuminated by a plane wave propagating along the \( \eta \) axis. (b) In the Born (or the Rytov) approximation, the modified sinogram is related to the scattered field (or the unwrapped complex phase). According to the generalized central-slice theorem, the 1D Fourier transform \( M(v_m, \phi) \) of the modified sinogram equals the 2D Fourier transform \( A(v_x, v_y) \) of the object function along a semicircle \( AOB \) that has a radius of \( v_0 \) and a center at \( v_y = -v_0 \).
\( U_s(v_m, \phi) \)

\[
\begin{align*}
&\left[ 2\pi^2 v_0^2 U_0 \exp(j2\pi v'l) \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\bar{r}) \\
&\times \exp[-j2\pi v_m \xi + (v' - v_0) \eta]]d\bar{r} & \text{if } |v_m| \leq v_0 \\
&0 & \text{if } |v_m| > v_0
\end{align*}
\]

(4)

where \( \eta = -r \sin(\phi - \theta) \) and \( v' \) is a function of \( v_m \), given by

\[
v' = \sqrt{v_0^2 - v_m^2}.
\]

\( \text{Rytov approximation. In an alternative approach to solving the Helmholtz equation, the scattered field } u_s \text{ is transformed into a complex phase function as } u_s(\xi, \phi) = u_i(\xi, \phi)[\exp[\phi_i(\xi, \phi)] - 1]. \) Substituting this expression into Eq. (2) and making the Rytov approximation by ignoring the term \([\nabla \phi_i(\xi, \phi) \nabla \phi_j(\xi, \phi)]\) yields a linear equation for the complex phase function \( \phi_j(\xi, \phi) \), which is given by

\[
\Psi_j(v_m, \phi) = \int_{-\infty}^{\infty} \phi_j(\xi, \phi) \exp(-j2\pi v_m \xi)dv_m.
\]

This equation has a form similar to that of the equation for \( u_i \) in the Born approximation, and hence an analytic relationship between \( \phi_j(\xi, \phi) \) and \( a(\bar{r}) \) can be derived from this equation. It should be noted that, although the derivation of the relationship between \( \phi_j(\xi, \phi) \) and \( a(\bar{r}) \) in the Rytov approximation is mathematically equivalent to the derivation of the relationship between \( u_i(\xi, \phi) \) and \( a(\bar{r}) \) in the Born approximation, the physical meaning and hence the validity in practical applications is different. Let \( \Psi_j(v_m, \phi) = \int_{-\infty}^{\infty} \phi_j(\xi, \phi) \exp(-j2\pi v_m \xi)dv_m \) be the 1D Fourier transform of \( \phi_j(\xi, \phi) \) with respect to \( \xi \) and \( v_m \). It can be shown that this relationship is given by

\[
\Psi_j(v_m, \phi) = \left[ 2\pi^2 v_0^2 U_0 \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\bar{r}) \\
\times \exp[-j2\pi v_m \xi + (v' - v_0) \eta]]d\bar{r} & \text{if } |v_m| \leq v_0 \\
0 & \text{if } |v_m| > v_0
\]

(7)

where \( \eta = -r \sin(\phi - \theta) \) and the function \( v' \) is given by Eq. (5).

\( \text{Modified sinogram. In both the Born and the Rytov approximations, analytic relationships between the Fourier transforms of a measurable function } (u_s \text{ and } \phi_s) \text{ and the object function } a(\bar{r}) \text{ are established. We notice that in both relationships } [\text{see Eqs. (4) and (7)}], \text{ the integrals that involve the object function } a(\bar{r}) \text{ are identical and are given by}

\[
M(v_m, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\bar{r}) \exp[-j2\pi v_m \xi + (v' - v_0) \eta]]d\bar{r}.
\]

(8)

This function can be interpreted as the 2D Fourier transform of the object function \( a(\bar{r}) \) along a semicircle oriented at an angle \( \phi \) in the 2D Fourier space of \( a(\bar{r}) \), as shown in Fig. 2(b), and it thus describes a generalized central-slice theorem that becomes the conventional central-slice theorem as \( v_0 \to \infty \). The physical meaning of \( v_0 \to \infty \) is that the incident radiation has high energy and travels in straight lines (like the x rays in CT).

We introduce a modified sinogram,

\[
m(\xi, \phi) = \int_{-\infty}^{\infty} M(v_m, \phi) \exp(j2\pi v_m \xi)dv_m,
\]

(9)

which is the 1D inverse Fourier transform of \( M(v_m, \phi) \) and was originally proposed for 2D SPECT with uniform attenuation. Introduction of the concepts of the modified sinogram \( m(\xi, \phi) \) and its 1D Fourier transform \( M(v_m, \phi) \) allows the development of our reconstruction theory for diffraction tomography that can be used in both the Born and the Rytov approximations.

It should be noticed from Eqs. (2) and (8) that the 1D Fourier transforms \( P(v_a, \phi_0) \) and \( M(v_m, \phi) \) of the ideal and the modified sinograms are not identical to each other when \( v_a = v_m \) and \( \phi_0 = \phi \). On the other hand, because both \( P(v_a, \phi_0) \) and \( M(v_m, \phi) \) are related to the 2D Fourier transform of the object function, one can find relationships between \( P(v_a, \phi_0) \) and \( M(v_m, \phi) \) that, in turn, depend on the relationship between the variables \( (v_a, \phi_0) \) and \( (v_m, \phi) \). We will derive the relationships between \( (v_a, \phi_0) \) and \( (v_m, \phi) \) in Appendix A by use of an approach that is alternative to and simpler than that described previously in the literature.

3. FOURIER ANALYSIS OF SINOGRAMS

In this section, we carry out the Fourier analysis of the ideal and the modified sinograms and derive the relationships between them.

A. Two-Dimensional Fourier Transform of the Ideal Sinogram

The result of the 2D Fourier transform of the ideal sinogram, which we denote by \( P_k(v_a) \), is given by

\[
P_k(v_a) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(v_a, \phi_0) \exp(-j2\pi v_a \xi) \\
\times \exp(-jk \phi_0) d\xi d\phi_0,
\]

(10)

where \( k \) is an integer. Note that the angular frequency index \( k \) is discrete because \( p(\xi, \phi_0) \) is periodic in \( \phi_0 \). Hence “the 2D Fourier transform” of the sinogram is, in fact, a combination of a 1D Fourier transform and a 1D Fourier series expansion. It can be shown that substitution of Eq. (2) into Eq. (10) yields

\[
P_k(v_a) = (-j)^k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\bar{r}) \exp(-jk \theta) J_k(2\pi v_a r)drd\theta,
\]

(11)

where \( J_k \) indicates the \( k \)-th order Bessel function of the first kind. Because \( J_k(x) = (-1)^k J_{-k}(-x) \), from Eq. (11),

\[
P_k(v_a) = (-1)^k P_k(-v_a)
\]

(12)

for all integers \( k \) and all real \( v_a \). This constraint on \( P_k(v_a) \) implies that the task of estimating the ideal sino-
gram is equivalent to that of estimating \( P_k(v_a) \) in only two quadrants of the \((v_n, k)\) Fourier space, which we take as \( v_a \gg 0 \). In 2D SPECT with uniform attenuation,\(^{34}\) we derived a similar constraint on \( P_k(v_a) \), which implies that we need to determine \( P_k(v_a) \) in only one quadrant of the \((v_n, k)\) Fourier space.\(^{34}\) The difference between the constraints in 2D SPECT and 2D diffraction tomography arises because the object function \( a(\bar{r}) \) is real in SPECT, whereas the object function \( a(\bar{r}) \) can be complex in 2D diffraction tomography.

**B. Two-Dimensional Fourier Transform of the Modified Sinogram**

The 2D Fourier transform of the modified sinogram is given by

\[
M_k(v_m) = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} m(\xi, \phi) \exp(-j2\pi v_m \xi) \times \exp(-jk \phi) d\xi d\phi.
\]  

(13)

Substituting Eqs. (9) and (8) into Eq. (13) and noting that \( \xi = r \cos(\phi - \theta) \) and \( \eta = -r \sin(\phi - \theta) \) yields

\[
M_k(v_m) = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} a(\bar{r}) \left( \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \right. \\
\times \exp[\mu' r \sin(\phi - \theta) - j[k \phi + 2\pi v_m] ] \times r \cos(\phi - \theta)] d\phi dr d\theta,
\]  

(14)

where \( v_m \) is the spatial frequency of the modified sinogram and where \( \mu' = j2\pi[(v_0^m - v_m^2)^{1/2} - v_0] \) is, for \( |v_m| \leq v_0 \), a purely imaginary function of \( v_m \).

Using the result that we derived before\(^{34}\) for the integral in bold parentheses of Eq. (14), we obtain an expression for \( M_k(v_m) \) in terms of the Bessel function of the first kind, given by

\[
M_k(v_m) = (-j)^k \left[ \frac{v_m + \mu}{(v_m^2 - \mu^2)^{1/2}} \right]^k \\
\times \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} a(\bar{r}) \exp(-jk \theta) \\
\times J_k[2\pi r(v_m^2 - \mu^2)^{1/2}] dr d\theta \quad \text{if} \quad v_m^2 \gg \mu^2,
\]  

(15)

where

\[
\mu = \frac{\mu'}{2\pi} = j[(v_0^2 - v_m^2)^{1/2} - v_0].
\]  

(16)

It can readily be shown that \( \mu^2 \geq v_a^2 \) is equivalent to \( |v_m| \leq v_0 \).

**C. Relationships between the Fourier Transforms**

A result with a form similar to that of Eq. (15) was obtained for 2D SPECT with uniform attenuation.\(^{34}\) However, \( \mu \) is a real constant equal to the ratio of the constant-attenuation coefficient \( \mu \) and \( 2\pi \) in that situation, whereas in diffraction tomography \( \mu \) is a purely imaginary function of the spatial frequency \( v_m \) of the modified sinogram for \( |v_m| \leq v_0 \).

Comparison of Eqs. (11) and (15) yields

\[
P_k(v_a) = \left[ \frac{(v_m^2 - v_a^2)^{1/2}}{v_m + v_a} \right] M_k(v_m) \quad \text{if} \quad |v_m| \leq v_0,
\]  

(17)

where the spatial frequency \( v_a \) of the ideal sinogram is related to the spatial frequency \( v_m \) of the modified sinogram by

\[
v_a^2 = v_m^2 - v_\mu^2.
\]  

(18)

The relationship between \( v_a \) and \( v_m \) was also obtained in Appendix A by use of an alternative approach.

Equations (17) and (18) provide a general relationship between the 2D Fourier transforms \( P_k(v_a) \) and \( M_k(v_m) \) of the ideal and modified sinograms, respectively. Therefore, this relationship can be used to estimate \( P_k(v_a) \) [and, hence, the ideal sinogram that is simply a 2D inverse Fourier transform of \( P_k(v_a) \)] from \( M_k(v_m) \), which can be obtained from the measurable quantities. The explicit forms of this general relationship in Eq. (17) are determined completely by the explicit relationship between \( v_a \) and \( v_m \) implied in Eq. (18). As shown in Appendix B, the relationship between \( P_k(v_a) \) and \( M_k(v_m) \) [Eq. (17)] can also be obtained by use of the generalized central-slice theorem and the relationship between \( (v_a, \phi_0) \) and \( (v_m, \phi_m, \phi_2)^{34} \).

As discussed above, we need to estimate \( P_k(v_a) \) only for \( v_a \geq 0 \) [see Eq. (12)]. Therefore, by using Eqs. (18) and (16), one can show that two different roots, \( v_{m1} \) and \( v_{m2} \), which are given by

\[
v_{m1} = -v_{m2} = v_m = v_{a}(1 - v_\mu^2/4v_a^2)^{1/2},
\]  

(19)

satisfy Eq. (18) for any given \( 0 \leq v_a \leq \sqrt{2} v_0 \). Note that \( v_a \) is assumed to be positive. Equation (19) gives an explicit and nonlinear relationship between \( v_{m1} \) (or \( v_{m2} \)) and \( v_a \), which is shown in Fig. 3.

For notational convenience in discussing the implications of Eq. (17), we introduce the function

\[
\gamma(v_m) = \frac{(v_m^2 - v_\mu^2)^{1/2}}{v_m + v_\mu}
\]  

(20)
and notice that, for $v_m^2 \geq v_n^2$ (or, equivalently, $|v_m| \leq v_0$),

$$
\gamma(-v_m) = -\gamma(v_m)^{-1}, \quad \gamma(v_m)^* = \gamma(v_m)^{-1}. \tag{21a}
$$

The complex function $\gamma(v_m)$ has a unit modulus because

$$
|\gamma(v_m)|^2 = \gamma(v_m)\gamma(v_m)^* = \gamma(v_m)\gamma(v_m)^{-1} = 1, \tag{21b}
$$

and hence the function $\gamma(v_m)$ can also be written as

$$
\gamma(v_m) = \exp[j \beta(v_m)], \tag{21c}
$$

where the phase $\beta(v_m)$ is generally a real function of $v_m$.

Therefore the 2D Fourier transform $P_k(v_n)$ of the ideal sinogram can be estimated from the 2D Fourier transform of the modified sinogram at $v_{m1} = v_m$; i.e.,

$$
P_k(v_n) = [\gamma(v_{m1})]^kM_k(v_{m1}) = [\gamma(v_m)]^kM_k(v_m), \tag{22}
$$

for $0 \leq v_n \leq \sqrt{2}v_0$, where $0 \leq v_m \leq v_0$ is given by Eq. (19).

On the other hand, $P_k(v_n)$ can also be estimated from the 2D Fourier transform of the modified sinogram at $v_{m2} = -v_m$; i.e.,

$$
P_k(v_n) = [\gamma(v_{m2})]^kM_k(v_{m2}) = (-1)^k[\gamma(v_m)]^{-k}M_k(-v_m) \tag{23}
$$

for $0 \leq v_n \leq \sqrt{2}v_0$, where $0 \leq v_m \leq v_0$ is again given by Eq. (19).

The physical origin of Eqs. (22) and (23) can be understood as follows. According to the generalized central-slice theorem, the 1D Fourier transform of the modified sinogram, $M(v_{m1}, \phi)$, is equal to the 2D Fourier transform of $a(\tau)$ evaluated along a semicircle, as shown in Fig. 2(b). As the projection angle $\phi$ is varied from 0 to $2\pi$, the two portions, OA and OB, of the semicircle AOB generate two distinct coverages of the Fourier space, as shown in Fig. 4. We assume $0 \leq v_m \leq v_0$ without losing generality. The 1D Fourier transform of the modified sinogram can be expressed as $M(v_m, \phi)$ and $M(-v_m, \phi)$ on the two segments, OA and OB, respectively, which can subsequently be employed for calculation of the 2D Fourier transforms, $M_k(v_m)$ and $M_k(-v_m)$, by use of $(v_m, \phi)$ and $(-v_m, \phi)$ in Eq. (14), respectively. We show in Appendix B that $M_k(v_m)$ and $M_k(-v_m)$ obtained in this way are related to $P_k(v_n)$ through Eqs. (22) and (23), respectively.

4. NOISE ANALYSIS

The 2D Fourier transform $P_k(v_n)$ of the ideal sinogram can be estimated by use of Eqs. (22) and (23) from knowledge of $M_k(v_m)$ at positive and negative spatial frequencies $v_m$. In the absence of noise, the two estimates of $P_k(v_n)$ are identical. However, as we will show below, the two estimates of $P_k(v_n)$ will differ in general, when noise is present in the measured data and hence in $M_k(v_m)$. In this section we explore the statistical relationship between outcomes of $M_k(v_m)$ and $M_k(-v_m)$.

From Eqs. (4), (7), and (8), we see that $M(v_m, \phi)$ can be expressed as

$$
M(v_m, \phi) = f(v_m)S(v_m, \phi), \tag{24}
$$

where $S(v_m, \phi) = \int_{-\pi}^{\pi} s(\xi, \phi) \exp(-j2\pi v_m \xi) d\xi$ is the 1D Fourier transform of the measured sinogram $s(\xi, \phi)$ with respect to $\xi$, specifically,

$$
S(v_m, \phi) = \begin{cases} U_s(v_m, \phi) & \text{for the Born approximation} \\ \Psi_s(v_m, \phi) & \text{for the Rylov approximation} \end{cases}, \tag{25}
$$

and where

$$
\begin{align*}
U_s(v_m, \phi) &= \frac{jv'}{2\pi^2 v_0^2 U_0} \exp(-j2\pi v' l) \quad \text{for the Born approximation} \\
\Psi_s(v_m, \phi) &= \frac{jv'}{2\pi^2 v_0^2 U_0} \exp(-j2\pi(v' - v_0) l) \quad \text{for the Rylov approximation} \tag{26}
\end{align*}
$$

Analysis for generally correlated noise. The measured diffraction sinogram contains noise and can be treated as a stochastic process $s(\xi, \phi)$ with a mean of $s(\xi, \phi)$.34,37 (Here and in the following, bold and lightface italic letters denote a stochastic process and its mean, respectively.) Let us assume that the autocovariance of $s(\xi, \phi)$ is given by $\text{covar}\{s(\xi, \phi), s(\xi', \phi')\}$.

Using Eq. (24) and the expression for the autocovariance of $s(\xi, \phi)$, one can express the autocovariance of $M_k(v_m)$ as

$$
\text{covar}\{M_k(v_m), M_k(v'_m)\}
\begin{align*}
&= \frac{f(v_m)f^*(v'_m)}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \text{covar}\{s(\xi, \phi), s(\xi', \phi')\} \\
&\quad \times \exp[-j(k \phi - k' \phi')] \\
&\quad \times \exp[-j2\pi(v_m \xi - v'_m \xi')] d\xi d\xi' d\phi d\phi'. \tag{27}
\end{align*}
$$

---

Fig. 4. As the projection angle $\phi$ varies from 0 to $2\pi$, the semicircle AOB rotates around the origin of the Fourier space of the object function, and two distinct coverages, (a) and (b), of the Fourier space are generated by the two segments OA and OB, respectively, of the semicircle AOB.
Specifically, the covariance between the outcomes of $M_k(v_m)$ at positive and negative $v_m \neq 0$ is given by

$$\text{covar}\{M_k^2(v_m), M_k^*(-v_m)\}$$

$$= \frac{|f(v_m)|^2}{4\pi^2} \int_0^{2\pi} \int_{\phi, \phi'=0}^{\infty} \int_{\xi, \xi'=-\infty}^{\infty} \text{covar}\{s(\xi, \phi), s(\xi', \phi')\} \times \exp[-j k (\phi - \phi')] \times \exp[-j 2 \pi v_m (\xi - \xi')] \delta \xi \delta \phi \delta \phi',$$

and the variances of $M_k(v_m)$ and $M_k(-v_m)$ are given, respectively, by

$$\tau_{k\pm}^2(v_m) = \text{var}\{M_k(v_m)\}$$

$$= \text{covar}\{M_k(v_m), M_k(v_m)\}$$

$$= \frac{|f(v_m)|^2}{4\pi^2} \int_0^{2\pi} \int_{\phi, \phi'=0}^{\infty} \int_{\xi, \xi'=-\infty}^{\infty} \text{covar}\{s(\xi, \phi), s(\xi', \phi')\} \times \exp[-j k (\phi - \phi')] \times \exp[-j 2 \pi v_m (\xi - \xi')] \delta \xi \delta \phi \delta \phi',$$

and

$$\tau_{k\pm}^2(-v_m) = \text{var}\{M_k(-v_m)\}$$

$$= \text{covar}\{M_k(-v_m), M_k(-v_m)\}$$

$$= \frac{|f(v_m)|^2}{4\pi^2} \int_0^{2\pi} \int_{\phi, \phi'=0}^{\infty} \int_{\xi, \xi'=-\infty}^{\infty} \text{covar}\{s(\xi, \phi), s(\xi', \phi')\} \times \exp[-j k (\phi - \phi')] \times \exp[j 2 \pi v_m (\xi - \xi')] \delta \xi \delta \phi \delta \phi',$$

(28a)

(29a)

It should be noted that $\tau_{k\pm}(v_m)$ and $\tau_{k\pm}(-v_m)$ may be different for generally correlated data noise. However, in some situations, as discussed below and in the literature,26,27 $\tau_{k\pm}(v_m)$ and $\tau_{k\mp}(v_m)$ are identical to each other.

**Analysis for uncorrelated or stationary noise.** If the noise in the measured data is uncorrelated,27 i.e.,

$$\text{covar}\{s(\xi, \phi), s(\xi', \phi')\} = \sigma^2(\xi, \phi) \delta(\xi - \xi') \delta(\phi - \phi'),$$

(30)

where $\delta(\cdot)$ is the Dirac function and where $\sigma^2(\xi, \phi)$ is the variance of data at $(\xi, \phi)$, the autocovariance between $M_k(v_m)$ and $M_k(-v_m)$ can be expressed as

$$\text{covar}\{M_k(v_m), M_k(-v_m)\}$$

$$= \frac{|f(v_m)|^2}{4\pi^2} \int_0^{2\pi} \int_{\xi, \xi'=-\infty}^{\infty} \sigma^2(\xi, \phi) \exp(-j 4 \pi v_m \xi) \delta \xi \delta \phi,$$

(31)

and the variances of $M_k(v_m)$ and $M_k(-v_m)$ are given by

$$\tau_{k\pm}^2(v_m) = \tau_{k\pm}^2(-v_m) = \sigma^2(v_m)$$

$$= \frac{|f(v_m)|^2}{4\pi^2} \int_0^{2\pi} \int_{\phi, \phi'=0}^{\infty} \int_{\xi, \xi'=-\infty}^{\infty} \sigma^2(\xi, \phi) \delta(\xi - \xi') \delta(\phi - \phi') \delta \xi \delta \phi \delta \phi' \delta \xi' \delta \phi' \delta \xi' \delta \phi',$$

(32)

Therefore the variances of $M_k(v_m)$ and $M_k(-v_m)$ in the data is uncorrelated.

As indicated in Eq. (31), $\text{covar}\{M_k(v_m), M_k(-v_m)\}$ is generally nonzero. Also, for $v_m \neq 0$, $\text{covar}\{M_k(v_m), M_k(-v_m)\}$ is less than $\text{var}\{M_k(v_m)\}$ [see Eqs. (31) and (32)]. This implies that the values of $M_k(v_m)$ and $M_k(-v_m)$ are not completely redundant and that the two estimates of $P_k(v_j)$ by use of Eqs. (22) and (23) are generally different in the presence of noise. The above observations suggest that there is complementary information in $M_k(v_m)$ and $M_k(-v_m)$, which can be used for reducing the variance in estimating $P_k(v_j)$.

An alternative and frequently used approach26,28 to modeling noise in the measured scattered data is as follows. The scattering object is assumed to be a 2D wide-sense-stationary stochastic process, $a(\vec{r})$. It has been shown26,28 that the scattered sinogram $s(\xi, \phi)$ of the plane-wave radiation from this scattering object $a(\vec{r})$ is also stationary in $\xi$ and is uncorrelated in $\phi$. Furthermore, if (a) the stochastic process $a(\vec{r})$ is statistically isotropic, i.e., the autocorrelation of $a(\vec{r})$ between $r_1$ and $r_2$, which depends only on $|r_1 - r_2|$, or if (b) there is no absorption in the scattering process so that the stochastic process $a(\vec{r})$ is real, then it can be shown that $\tau_{k\pm}^2(v_m) = \tau_{k\pm}^2(-v_m)$. In this work, for simplicity, we will call the noise that satisfies conditions (a) and/or (b) stationary noise. Therefore, for stationary noise, the variances $\tau_{k\pm}^2(v_m)$ and $\tau_{k\pm}^2(-v_m)$ of $M_k(v_m)$ and $M_k(-v_m)$ are also identical.

**Global variance analysis.** Using the orthonormal properties of the basis functions $\{\exp[j k \theta d \xi (2\pi v_m)]\}$ in Eq. (11), one can obtain an expression for the estimate of the object function in terms of the estimate $P_k(v_m)$:

$$a(\vec{r}) = 2 \pi \sum_{k=-\infty}^{\infty} j_k \int_{v_m=0}^{\infty} P_k(v_m) \exp[j k \theta]$$

$$\times J_k(2\pi v_a \rho) v_a d v_a.$$

(33)

From Eq. (33), one can obtain an expression for the global variance of the reconstructed object function, which is given by24

$$\int_{\theta=0}^{2\pi} \int_{\rho=0}^{\infty} \text{var}\{a(\vec{r}, \theta)\} r d \theta$$

$$= 2 \pi \sum_{k=-\infty}^{\infty} \int_{v_m=0}^{\infty} \text{var}\{P_k(v_m)\} v_a d v_a.$$

(34)

Therefore Eq. (34) implies that the object function with a lower global variance can be obtained when it is reconstructed from the estimated $P_k(v_m)$ with a reduced variance. Because the local variance $\text{var}\{a(\vec{r})\}$ is nonnegative for all $r$, a lower global variance suggests, in general, lower local variances in the reconstructed object function. In the following section we discuss the strategy for obtaining an estimate of $P_k(v_m)$ with reduced variance.
5. ESTIMATION OF THE IDEAL SINOGRAM

As discussed above, in the presence of noise the estimates of \( P_k(v_a) \) obtained from \( M_k(v_m) \) [Eq. (22)] and \( M_k(-v_m) \) [Eq. (23)] contain complementary information that can be used to reduce the variance of the final estimate of \( P_k(v_a) \). Thus the strategy of estimating \( P_k(v_a) \) from knowledge of both \( M_k(v_m) \) and \( M_k(-v_m) \) appears to have an advantage over use of knowledge of either \( M_k(v_m) \) or \( M_k(-v_m) \) alone. This observation has been verified in 2D SPECT with uniform attenuation.\(^{24,30}\)

A. Linear Combination of Estimates

The 2D Fourier transform \( P_k(v_a) \) of the ideal sinogram can be estimated in a variety of linear and nonlinear combinations of \( M_k(v_m) \) or \( M_k(-v_m) \). However, a general linear combination of the estimates provided by Eqs. (22) and (23) is a natural and mathematically appealing approach and can be expressed as

\[
P_k(v_a) = \omega \gamma^5 M_k(v_m) + (1 - \omega)(-1)^k \gamma^{-5} M_k(-v_m),
\]

where the combination coefficient \( \omega = R + jI \) is generally complex, with a real part \( R \) and an imaginary part \( I \), and where the function \( \gamma \) is given by Eq. (20). From Eqs. (22) and (23), one can show that the final estimate of \( P_k(v_a) \) in Eq. (35) is a bias-free estimate for any values of the combination coefficient \( \omega \). Therefore the introduction of \( \omega \) provides an opportunity for a bias-free reduction of the variance of the final estimate of \( P_k(v_a) \).

Analysis for generally correlated noise. It can be shown that the variance of the final estimate of \( P_k(v_a) \) is given by

\[
\text{var}\{P_k(v_a)\} = \tau_{k+}(v_m) \tau_{k-}(v_m) \times \frac{[t_+ + t_- - 2(-1)^k \rho_k^{\prime\prime}(v_m)]}{(R^2 + I^2)} \times \frac{2(1 + t_+ t_-)}{R - (-1)^k \rho_k^{\prime\prime}(v_m) I + t_-},
\]

where

\[
t_+ = 1/t_- = \frac{\tau_{k+}(v_m)}{\tau_{k-}(v_m)}
\]

and where the correlation coefficient \( \rho_k(v_m) = \rho_k^{\prime\prime}(v_m) + j \rho_k^{\prime\prime}(v_m) \) is generally a complex function comprising real and imaginary parts that are given by

\[
\rho_k^{\prime\prime}(v_m) = \text{Re}\left[\gamma^2 \text{var}\{M_k(v_m), M_k(-v_m)\} \frac{\tau_{k+}(v_m)}{\tau_{k-}(v_m)} \tau_{k-}(v_m) \right]
\]

and

\[
\rho_k^{\prime\prime}(v_m) = \text{Im}\left[\gamma^2 \text{var}\{M_k(v_m), M_k(-v_m)\} \frac{\tau_{k+}(v_m)}{\tau_{k-}(v_m)} \tau_{k-}(v_m) \right]
\]

respectively, and where \( \text{Re} \) and \( \text{Im} \) denote the real and imaginary parts of a complex function.

The optimal choice of the combination coefficient \( \omega \) that minimizes the variance of \( P_k(v_a) \) in Eq. (36) (and thus the global variance [see Eq. (34)]) in the reconstructed object function) can be obtained by solving two equations resulting from

\[
\frac{\partial \text{var}\{P_k(v_a)\}}{\partial R} = 0 \quad \text{and} \quad \frac{\partial \text{var}\{P_k(v_a)\}}{\partial I} = 0.
\]

Therefore the real and imaginary parts of the optimal combination coefficient \( \omega_{\text{op}} \) are given by

\[
R_{\text{op}} = \frac{t_+ + t_- - 2(-1)^k \rho_k^{\prime\prime}(v_m)}{t_+ + t_- - 2(-1)^k \rho_k^{\prime\prime}(v_m)},
\]

and

\[
I_{\text{op}} = \frac{(-1)^k \rho_k^{\prime\prime}(v_m)}{t_+ + t_- - 2(-1)^k \rho_k^{\prime\prime}(v_m)},
\]

respectively, and the resulting minimum variance of \( P_k(v_a) \) is given by

\[
\text{var}\{P_k(v_a)\}_{\text{min}} = \tau_{k+}(v_m) \tau_{k-}(v_m) \times \frac{1 - |\rho_k(v_m)|^2}{t_+ + t_- - 2(-1)^k \rho_k^{\prime\prime}(v_m)},
\]

where \( |\rho_k(v_m)|^2 = |\rho_k^{\prime\prime}(v_m)|^2 + |\rho_k^{\prime\prime}(v_m)|^2 \). The variance value in Eq. (39) is a true minimum because

\[
\frac{\partial^2 \text{var}\{P_k(v_a)\}}{\partial R^2} \geq \left[ \frac{\partial^2 \text{var}\{P_k(v_a)\}}{\partial R \partial I} \right]^2
\]

when evaluated at \( \omega = \omega_{\text{op}} \).

The dependence of \( \omega_{\text{op}} \) in Eq. (38) on the second-order statistics of the noise in the data indicates that the optimal combination coefficient \( \omega_{\text{op}} \) is not, in general, obtainable without prior knowledge of the data noise properties such as \( \tau_{k+}(v_m) \), \( \tau_{k-}(v_m) \), and \( \rho_k(v_m) \). However, as is the case in some practical situations, the second-order statistics of the data may be estimated from measurements.\(^{26}\) Therefore, by using the information on the second-order statistics in Eqs. (38) and (39), one can obtain the optimal estimate for the 2D Fourier transform of the ideal sinogram and consequently a reconstructed object function with a minimum global variance. On the other hand, it is always desirable to obtain the optimal combination coefficient without prior knowledge of the second-order statistics in the data. As will be discussed below, this may be possible for uncorrelated data noise or stationary noise. Therefore from this point on the focus will be on a discussion of uncorrelated and/or stationary data noise. However, the concepts, strategy, and analysis can readily be extended to generally correlated data noise.

Analysis for uncorrelated or stationary noise. For uncorrelated noise, using the results of Eqs. (31) and (32) in Eqs. (38) and (39) yields the optimal combination coefficient and the minimum variance of \( P_k(v_a) \), which are given by

\[
R_{\text{op}} = \frac{1}{2} \quad \text{and} \quad I_{\text{op}} = \frac{1}{2} \frac{(-1)^k \rho_k^{\prime\prime}(v_m)}{1 - (-1)^k \rho_k^{\prime\prime}(v_m)}
\]

and by
The key to obtaining the much simpler form in Eq. (40) compared with that in Eq. (38) is due to the fact that the variances \( \tau_{v}^{2}(v_{m}) \) and \( \tau_{v}^{2}(v_{n}) \) of \( M_{k}(v_{m}) \) at \( v_{m} \) and \( v_{n} \) are identical for the uncorrelated noise [see Eq. (32)]. As discussed previously, for stationary noise, one also has \( \tau_{k}^{2}(v_{n}) = \tau_{k}^{2}(v_{m}) \). Therefore the equations for the optimal combination coefficient and the minimum variance of \( P_{k}(v_{n}) \) have forms identical to that of Eqs. (40) and (41) when \( P_{k}(v_{n}) \) is estimated from the scattered data with stationary noise. The important implication of Eq. (40) suggests that the real part of the optimal combination coefficient \( \omega_{op} \) is obtainable without prior knowledge of the second-order statistics of the noise as long as the noise is uncorrelated or stationary, as discussed above.

In the reconstruction theory that was developed for 2D SPECT with uniform attenuation, the value of \( \omega \) was restricted to being real (i.e., \( \omega = R \)), and \( R_{op} \) was not obtainable because it requires prior knowledge of the properties of the data noise such as the correlation coefficient \( \rho_{k}(v_{m}) \). An infinite class of methods for estimating \( P_{k}(v_{n}) \) can be obtained from Eq. (35) when the value of \( \omega \) is restricted to being real, i.e., the values of \( \omega \) are restricted to being in \([0, 1]\) on the real axis of a complex plane, as shown in Fig. 5. For example, when \( \omega = 1 \), only \( M_{k}(v_{n}) \) [see Eq. (22)] is used for estimating \( P_{k}(v_{n}) \). On the other hand, when \( \omega = 0 \), only \( M_{k}(-v_{m}) \) [see Eq. (23)] is used for estimating \( P_{k}(v_{n}) \). Because the optimal coefficient is \( \omega_{op} = R_{op} = \frac{1}{2} \) in this situation, the estimation methods in the class with the real part of \( \omega \neq \frac{1}{2} \) always yield estimates of \( P_{k}(v_{n}) \) with higher variances, and thus they are of lesser interest.

However, the choice of \( R_{op} = \frac{1}{2} \) as the combination coefficient is suboptimal because, in addition to \( R_{op} = \frac{1}{2} \), the true optimal combination coefficient \( \omega_{op} \) also has an imaginary part \( I_{op} \), given by Eq. (40). The determination of \( I_{op} \) generally requires prior knowledge of the properties of the data noise, such as the correlation coefficient \( \rho_{k}(v_{m}) \).

\[
\text{var}(P_{k}(v_{n}))_{\text{min}} = \frac{\tau_{n}^{2}(v_{m}) - |\rho_{k}(v_{m})|^{2}}{2(1 - (-1)^{k})}, \quad (41)
\]

respectively.

The function \( \gamma \) is given by Eq. (20), and where the index \( n \) may have any real value. By use of the properties of the \( \gamma \) function in Eq. (21), it can be shown that

\[
\omega = \omega_{k}^{(n)}(v_{m}) = \frac{\gamma}{\gamma^{-nk} + \gamma^{-nk}}, \quad (42)
\]

where the function \( \gamma \) is given by Eq. (20), and where the index \( n \) may have any real value. By use of the properties of the \( \gamma \) function in Eq. (21), it can be shown that

\[
\omega_{k}^{(n)}(v_{m}) = \omega_{k}^{(n)}(-v_{m}) = 1 - \omega_{k}^{(n)}(v_{m}), \quad (43a)
\]

\[
\omega_{k}^{(n)}(v_{m}) = \frac{1}{2} - j\frac{1}{2} \tan(nk\beta(v_{m})). \quad (43b)
\]

We choose the form in Eq. (42) for \( \omega_{k}^{(n)}(v_{m}) \) for the following three reasons. (1) This form was used in and had important implications for our previous work on image reconstruction in SPECT with uniform attenuation. (2) From the analysis in Section 6, one can see that, in order to establish a relationship between the ideal-sinogram estimation and GFBPP methods, the combination coefficient \( \omega_{k}(v_{m}) \) must satisfy \( \omega_{k}(v_{m}) = 1 - \omega_{k}(v_{m}) \). As shown in Eq. (43a), \( \omega_{k}^{(n)}(v_{m}) \) certainly satisfies this condition. Other functional forms for \( \omega_{k}(v_{m}) \) that satisfy this condition can also be chosen. (3) Most important, we notice that \( \text{Re}[\omega_{k}^{(n)}(v_{m})] = \frac{1}{2} \) for any real value of \( n \). Thus \( \omega_{k}^{(n)}(v_{m}) \) is also a point on the same vertical line on which the optimal combination coefficient \( \omega_{op} \) [see Eq. (40)] is located, as shown in Fig. 5. The location of \( \omega_{k}^{(n)}(v_{m}) \) on this vertical line moves as the value of \( n \) changes. Therefore the introduction of this form for \( \omega_{k}^{(n)}(v_{m}) \) provides a potential opportunity to approach \( \omega_{op} \) by selection of appropriate values of \( n \) in Eq. (42). Indeed, it is worthwhile to investigate analytically as well as numerically the optimal value of \( n \) that may generate a \( \omega_{k}^{(n)}(v_{m}) \) that is closest to \( \omega_{op} \). However, such an investigation is beyond the scope of the current work and is an area of ongoing research.

Substitution of \( \omega_{k}^{(n)}(v_{m}) \) into Eq. (35) yields

\[
P_{k}^{(n)}(v_{m}) = \omega_{k}^{(n)}(v_{m})\gamma^{k}M_{k}(v_{m}) + \omega_{k}^{(n)}(-v_{m})(-1)^{k} \times \gamma^{-k}M_{k}(-v_{m})
\]

\[
= \gamma^{-n}M_{k}(v_{m})
\]

\[
= \gamma^{-nk} + \frac{\gamma^{-nk} M_{k}(v_{m})}{\gamma^{-nk} + \gamma^{-nk}},
\]

\[
+ (-1)^{k} \frac{\gamma^{-nk} M_{k}(v_{m})}{\gamma^{-nk} + \gamma^{-nk}} + (-1)^{k} \frac{\gamma^{-nk} M_{k}(v_{m})}{\gamma^{-nk} + \gamma^{-nk}}.
\]

In effect, the choice of a particular value of the index \( n \) in Eq. (42) provides a particular combination coefficient and thus a particular linear estimation method [Eq. (44)] for estimating \( P_{k}(v_{n}) \). A superscript has been added to \( \omega_{k}^{(n)}(v_{m}) \) and \( P_{k}^{(n)}(v_{n}) \) to indicate that they are obtained by use of the index \( n \) in Eqs. (42) and (44). Therefore Eq.
(44) provides another infinite class of linear estimation methods for estimating $P_k(v_n)$ (or, equivalently, the ideal sinogram) because in principle the index $n$ can be assigned any real value. These estimation methods yield identical $P_k(v_n)$ in the absence of noise, but in general they respond to noise differently.

6. RELATIONSHIP BETWEEN SINOGRAm ESTIMATION AND GFBPP METHODS

In this section we discuss the reconstruction approaches by use of the expectation values of the Fourier transforms of the ideal and the modified sinograms. The object function $a(\tilde{r})$ can be reconstructed from the 1D Fourier transform $M(v_m, \phi)$ of the modified sinogram by use of a variety of reconstruction techniques. According to the generalized central-slice theorem [Eq. (8)], $M(v_m, \phi)$ is equal to the 2D Fourier transform of $a(\tilde{r})$ along a semi-circle, as shown in Fig. 2(b). A reconstruction technique was developed that converts the 2D Fourier transform of $a(\tilde{r})$ on the semicircle grids onto rectangular grids so that the 2D inverse fast Fourier transform can be used to yield $a(\tilde{r})$. Another reconstruction technique was proposed for converting the 2D Fourier transform of $a(\tilde{r})$ on the semicircle grids onto polar grids to produce the 1D Fourier transform of the ideal sinogram, from which the FBPJ and other conventional reconstruction methods can be used for reconstruction of $a(\tilde{r})$.

The object function $a(\tilde{r})$ can be reconstructed from the 2D Fourier transform $P_k(v_n)$ of the ideal sinogram estimated from the modified sinogram. For example, the object function $a(\tilde{r})$ can be reconstructed by use of $P_k(v_n)$ in Eq. (33). It can also be obtained by use of the FBPJ or other conventional methods on the 1D Fourier transform of the ideal sinogram that can readily be calculated from $P_k(v_n)$. Third, the 2D Fourier transform of $a(\tilde{r})$ on rectangular grids can be calculated from $P_k(v_n)$, from which $a(\tilde{r})$ can be reconstructed by use of the 2D fast Fourier transform.

A. Filtered Backpropagation Method

The FBPP method is an alternative to the reconstruction techniques discussed above. This technique can be expressed as

$$a_{\text{FBPP}}(\tilde{r}) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} P_k(v_n) \frac{v_0}{v} M(v_m, \phi) \times \exp(j2\pi v_m \xi + 2\pi v_m \eta) dv_m d\phi,$$  

where $M(v_m, \phi) = \sum_{2\pi} M_k(v_m) \exp(jk \phi)$ and where a subscript FBPP is added to $a(\tilde{r})$ to indicate that the object function is reconstructed by use of the FBPP method. Unlike the distance-independent backprojection step in the FBPJ method, the backpropagation step in the FBPP method is distance dependent (i.e., $\eta$ dependent) because of the factor $\exp(2\pi v_m \eta)$. The form of the FBPP method is very similar to that of the Tretiak–Metz method for 2D SPECT with uniform attenuation. However, the distance-dependent factor $\exp(2\pi v_m \eta)$ in the FBPP method also depends on $v_m$, because $v_m$ (see Eq. (16)) is a function of $v_m$, whereas the distance-dependent factor in the Tretiak–Metz method is independent of $v_m$.

B. Generalized Filtered Backpropagation Method

We now propose a generalization of the FBPP method that is characterized by the index $n$ as used in Eq. (45) and that is expressed as

$$a_{\text{FBPP}}^{(n)}(\tilde{r}) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} P_k(v_n) \frac{v_0}{v} M^{(n)}(v_m, \phi) \times \exp(j2\pi v_m \xi + 2\pi v_m \eta) dv_m d\phi,$$  

where

$$M^{(n)}(v_m, \phi) = \sum_{k=-\infty}^{\infty} 2\omega_k^{(n)}(v_m) M_k(v_m) \exp(jk \phi),$$  

and where $\omega_k^{(n)}(v_m)$ is given by Eq. (42). Because Eq. (47) is the inverse Fourier series expansion of the product of $2\omega_k^{(n)}(v_m)$ and $M_k(v_m)$ in the angular frequency space, one can also interpret $M^{(n)}(v_m, \phi)$ as a convolution in the angular space between $M(v_m, \phi)$ and a convolver $f^{(n)}(v_m, \phi)$, which is given by

$$f^{(n)}(v_m, \phi) = \sum_{k=-\infty}^{\infty} 2\omega_k^{(n)}(v_m) \exp(jk \phi).$$

Therefore the GFBPP method in Eq. (46) has exactly the same form as that for the FBPJ method in Eq. (45), except that $M(v_m, \phi)$ in Eq. (45) is replaced by its convolved version, $M^{(n)}(v_m, \phi)$.

Each choice of the value of the index $n$ provides a particular method of the FBPP type. For example, one can obtain the FBPJ method in Eq. (45) by choosing $n = 0$ in Eq. (46). Because $n$ can be assigned any real value, Eq. (46) thus provides an infinite class of methods of the FBPP type. In the absence of data noise and with perfect sampling, the methods in Eq. (46) with different values of $n$ yield identical reconstructions of $a(\tilde{r})$. However, these methods generally respond to and propagate data noise differently.

C. Connection between Sinogram Estimation and GFBPP Methods

The object function can be obtained by using the estimate $P_k^{(n)}(v_n)$ in Eq. (33); i.e.,

$$a^{(n)}(\tilde{r}) = 2\pi \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} P_k^{(n)}(v_n) \times \exp(jk \theta) J_k(2\pi v_n r) v_n dv_n.$$

Substituting Eq. (44) into (49) and noting that $0 \leq v_n \leq \sqrt{2\pi v_0}$ and that $\omega_k^{(n)}(v_m) = 1 - \omega_k^{(n)}(v_m)$ yield

$$a^{(n)}(\tilde{r}) = 2\pi \sum_{k=-\infty}^{\infty} \int_{0}^{\infty} \left( \sum_{2\pi} \omega_k^{(n)}(v_m) \right)^k M_k(v_m) \times \exp(jk \theta) J_k(2\pi v_n r) v_n dv_n.$$  

Using the relationship between $v_n$ and $v_m$ in Eq. (18), one can show that, for $0 \leq v_m \leq v_0$, $v_n dv_n = \left( \frac{v_0}{v'} \right) v' dv_m$. Substitution of this result into Eq. (50) yields
It can be shown that the integrals in brackets in the first and second terms in Eq. (52) are given by
\[ j^k \gamma^\nu \exp(jk \theta) J_0(2\pi(v_m^2 - v^2)^{1/2}) \] and \[ j^k(-1)^\nu \gamma^\nu \times \exp(jk \theta) J_0(2\pi(v_m^2 - v^2)^{1/2}) \], respectively. [We have used \( \xi = r \cos(\phi - \theta) \) and \( \eta = -r \sin(\phi - \theta) \) in Eq. (52).] Therefore
\[
a^{(n)}(\vec{r}) = 2\pi \sum_{k=-\infty}^{\infty} j^k \exp(jk \theta) \times \int_{v_m=0}^{v_0} v_0 \nu \left\{ \omega_k^{(n)}(v_m) \gamma^k M_k(v_m) + \omega_k^{(n)}(-v_m)(-1)^k \gamma^k M_k(-v_m) \right\} \nu v_m du_m, \tag{53}
\]
A comparison of Eqs. (51) and (53) indicates that
\[
a^{(n)}(\vec{r}) = a_{\text{FBPP}}^{(n)}(\vec{r}). \tag{54}
\]
Therefore, in the absence of noise, the estimation method [Eq. (44)] for the 2D Fourier transform of the ideal sinogram and the GBPP method with the same value of the \( n \) are identical. Each estimation method with a particular \( n \) hence has a counterpart in the generalized FBPP method with the same value of \( n \). For example, the FBPP method proposed by Devaney\(^\text{25}\) has a counterpart in the ideal-sinogram estimation method with \( n = 0 \) in Eq. (44) and thus with a zero imaginary part of the combination coefficient. Because \( I_{\text{op}} \) in Eq. (40) is generally nonzero, this ideal-sinogram estimation method that corresponds to the FBPP method by Devaney\(^\text{25}\) is only statistically suboptimal.

7. DISCUSSION

In this paper we have studied the reconstruction problem of diffraction tomographic images from scattered data when plane-wave radiation is used to illuminate the scattering object. We established two relationships between the 2D Fourier transforms of the ideal sinogram \( P_k(v_m) \) and the modified sinogram \( M_k(v_m) \) that can be obtained from the measured scattered data. One can use the two relationships to calculate, from the scattered data, \( P_k(v_m) \) (or, equivalently, the ideal sinogram), which can be employed subsequently for reconstruction of the image of the scattering object by use of a wide variety of fast, reliable, and easy-to-implement reconstruction techniques such as the FBPP technique.

Noise analysis suggests that the two estimates of \( P_k(v_m) \) obtained by use of the two relationships are not entirely redundant statistically and contain complementary information. It was proposed to make use of such information by suggesting a general linear approach of combining the two estimates for a bias-free reduction of the variance in the reconstructed object function. From this perspective, we derived an optimal combination coefficient \( \omega_{\text{op}} \) [Eq. (38)] that yields a final estimate of \( P_k(v_m) \) with a minimum variance [Eq. (39)] when estimated from scattered data with generally correlated noise. The use of \( \omega_{\text{op}} \) requires prior knowledge of the second-order statistics of the scattered data [as shown in Eq. (38)], which are generally unknown (although, in some circumstances, the second-order statistics of the data noise may be estimated from the data\(^\text{26}\)).

In realistic diffraction tomography experiments, the scattered data often contain uncorrelated or stationary noise. In these situations we found that the real part of \( \omega_{\text{op}} \) is a constant, \( \frac{1}{2} \), and that only the imaginary part of \( \omega_{\text{op}} \) is dependent on the statistical properties of the data. In order to obtain the combination coefficient that may approximate the optimal coefficient \( \omega_{\text{op}} \) without prior knowledge of the statistics of the data, a combination coefficient \( \omega_k^{(n)}(v_m) \) was suggested that is characterized by an index number \( n \) [see Eq. (42)], which can assign any real value. As shown in Eq. (43b) and in Fig. 5, \( \omega_k^{(n)}(v_m) \) has a real part identical to that of \( \omega_{\text{op}} \) and different imaginary parts for different values of \( n \). Because each value of \( n \) yields a particular combination coefficient
and hence a particular estimation method for the ideal sinogram, in effect an infinite class of ideal-sinogram estimation methods is introduced.

The FBPP method is an alternative reconstruction approach to the ideal-sinogram estimation method. In this paper we proposed an infinite class of GFBPP methods [see Eq. (46)] that are again characterized by an index number $n$, which can be assigned any real values. The class of GFBPP methods includes the widely used FBPP method as a special member with $n = 0$. More important, the index number $n$ for each GFBPP method is exactly the same as that for the ideal-sinogram estimation method [see Eq. (44)], and hence each GFBPP method has a counterpart in the ideal-sinogram estimation method.

In addition, it was determined that the ideal-sinogram estimation method (with $n = 0$) that corresponds to the FBPP method by Devaney $^{25}$ is statistically suboptimal. However, future analytical and numerical investigations are needed to identify the member (i.e., the value of $n$) of this infinite class of ideal-sinogram estimation methods that is the closest to the optimal solution.

Object functions can be estimated by use of conventional reconstruction techniques from the ideal sinograms estimated with the estimation methods for different values of $n$, and they can also be estimated by use of the GFBPP methods with different values of $n$. In the absence of noise, these reconstructed object functions are identical, but they will appear different in the presence of noise because different combinations of ideal-sinogram estimation methods with conventional reconstruction techniques and different GFBPP methods propagate noise differently. Numerical implementation and study of the noise properties of the ideal-sinogram estimation methods and the GFBPP methods is currently under way and will be reported elsewhere.

The finite sampling that is employed in both data acquisition and reconstruction unavoidably affects the quantitative accuracy as well as the noise characteristics in the reconstructed object function. The effect of finite sampling on the reconstructed object function was not considered analytically in the theory reported here. However, such an effect, as well as the noise properties in the reconstructed object functions, can be explored and compared numerically for the proposed methods. How to incorporate the effect of finite sampling into the theory presented here and how and to what extent finite sampling affects the results are unclear.

In 2D SPECT with uniform attenuation, a combination coefficient similar to $\omega_{n}^{\text{FBPP}}(\nu_{m})$ [Eq. (38)] was used for ideal-sinogram estimation. It was shown $^{35}$ that the image reconstructed from the ideal sinogram estimated with $n = 2$ has a lower global image variance than do the images reconstructed from the ideal sinogram estimated with any other values of $n$. In diffraction tomography it is unclear so far what value of $n$ provides a combination coefficient $\omega_{n}^{\text{FBPP}}(\nu_{m})$ that is closest to the value of $\omega_{\text{opt}}$. This possible optimal value of $n$ deserves further analytical as well as numerical investigation.

In this paper we explored only a general linear combination of the two estimates of $P_{k}(\nu_{m})$ in an attempt to obtain a final estimate of $P_{k}(\nu_{m})$ with reduced variance. However, other nonlinear combinations of these two estimates can also be used and have not been investigated. In addition, we used a combination coefficient $\omega_{n}^{\text{FBPP}}(\nu_{m})$ that has the specific form of Eq. (42). One can certainly use other functional forms for $\omega_{n}^{\text{FBPP}}(\nu_{m})$ and can develop other types of GFBPP methods as long as the condition $\omega_{n}^{\text{FBPP}}(\nu_{m}) = 1 - \omega_{n}^{\text{FBPP}}(\nu_{m})$ is satisfied.

The theory developed in this paper is for reconstruction of the scatterer distribution in a 2D object that is illuminated with plane-wave radiation, and it is valid only under the Born or the Rytov approximation. How to extend the strategy and theory in this work to the higher-order Born or Rytov approximation $^{40}$ to non-plane-wave radiation $^{41}$ and to 3D diffraction tomography remains to be explored.

**APPENDIX A: AN ALTERNATIVE APPROACH FOR DERIVATION OF THE RELATIONSHIPS BETWEEN $(\nu_{a}, \phi_{b})$ AND $(\nu_{m}, \phi)$**

The relationships between $(\nu_{a}, \phi_{b})$ and $(\nu_{m}, \phi)$ have been derived previously. $^{24}$ In this appendix we shall use an alternative approach to derive the same relationships.

According to the central-slice theorem [see Eq. (2)], the 1D Fourier transform $P(\nu_{m}, \phi)$ of the ideal sinogram is given by

$$P(\nu_{a}, \phi_{0}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\nu_{m}) \exp[-j2\pi\nu_{a}r \cos(\phi_{0} - \theta)] \, d\nu_{m} \, d\nu_{a}.$$  \hspace{1cm} (A1)

Also, according to the generalized central-slice theorem [see Eq. (8)], the 1D Fourier transform $M(\nu_{m}, \phi)$ of the modified sinogram can be expressed as

$$M(\nu_{m}, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\nu_{m}) \exp[-j2\pi \nu_{m} \xi + (\nu' - \nu_{0}) \eta] \, d\nu_{m} \, d\nu_{a}.$$  \hspace{1cm} (A2)

where $\nu' = \left(\nu_{m}^{2} - \nu_{a}^{2}\right)^{1/2}$.

We rewrite Eq. (A2) as

$$M(\nu_{m}, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\nu_{m}) \exp\left[-j2\pi \left(\nu_{m}^{2} + (\nu' - \nu_{0})^{2}\right)^{1/2}\right] \times \left\{ \frac{\nu_{m}}{\left[\nu_{m}^{2} + (\nu' - \nu_{0})^{2}\right]^{1/2}} \xi + \frac{(\nu' - \nu_{0})}{\left[\nu_{m}^{2} + (\nu' - \nu_{0})^{2}\right]^{1/2}} \eta \right\} \, d\nu_{m} \, d\nu_{a}.$$  \hspace{1cm} (A3)

Defining

$$\cos \phi' = \frac{\nu_{m}}{\left[\nu_{m}^{2} + (\nu' - \nu_{0})^{2}\right]^{1/2}}$$

and

$$\sin \phi' = \frac{(\nu' - \nu_{0})}{\left[\nu_{m}^{2} + (\nu' - \nu_{0})^{2}\right]^{1/2}}$$  \hspace{1cm} (A4)
and noticing $\xi = r \cos(\phi - \theta)$ and $\eta = r \sin(\phi - \theta)$ in Eq. (A3) yields

$$M(v_m, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\hat{r}) \times \exp\{-j2\pi[v_m^2 + (v' - v_0)^2]^{1/2}\hat{r} \times [\cos(\phi' \cos(\phi - \theta)) - \sin(\phi' \sin(\phi - \theta))])d\hat{r}$$

\begin{align*}
M(v_m, \phi) & = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\hat{r}) \times \exp\{-j2\pi[v_m^2 + (v' - v_0)^2]^{1/2}\hat{r} \times \cos(\phi + \phi' - \theta))d\hat{r}. (A5)\\

A comparison of Eqs. (A1) and (A5) indicates that the 1D Fourier transforms of the ideal and modified sinograms are identical if we have

$$v_a = [v_m^2 + (v' - v_0)^2]^{1/2} \quad \text{and} \quad \phi_0 = \phi + \phi'. \quad (A6)$$

For a given $0 < v_a < \sqrt{2}v_0$ there are two roots, $v_m$ and $-v_m$, that satisfy Eq. (A6), where $v_m$ is given by

$$v_m = v_a(1 - v_a^2/4v_0^2)^{1/2}. \quad (A7)$$

From Eq. (A4) we see that, as a function of $v_m$, $\phi'(v_m) = \pi - \phi'(v_m)$. Using this observation and Eqs. (A7) and (A4) in (A6) yields

$$\phi = \begin{cases} 
\phi_0 + \arcsin(v_a/2v_0) & \text{for } v_m \\
\phi_0 - \arcsin(v_a/2v_0) - \pi & \text{for } -v_m, \end{cases} \quad (A8a)$$

and

$$P(v_a, \phi_0) = M(v_m, \phi), \quad (A9a)$$

where $\phi$ is given by Eq. (A8a), and

$$P(v_a, \phi_0) = M(-v_m, \phi), \quad (A9b)$$

where $\phi$ is given by Eq. (A8b).

The results in Eqs. (A7)–(A9) were obtained also by Pan and Kak$^{24}$ by consideration of the geometric relationships between $(v_a, \phi_0)$ and $(v_m, \phi)$. The relationships in Eq. (A9) show that $M(v_m, \phi)$ and $M(-v_m, \phi)$ are equal to the 2D Fourier transform of the object function on the segments $OA$ and $OB$ of the semicircle, respectively, as shown in Fig. 4.

**APPENDIX B: AN ALTERNATIVE APPROACH FOR DERIVATION OF THE RELATIONSHIPS BETWEEN $P_k(v_a)$ AND $M_k(v_m)$**

In Section 3 we obtained the relationships between $P_k(v_a)$ and $M_k(v_m)$ [Eqs. (22) and (23)] by carrying out the integral in the curly brackets of Eq. (14). In this appendix we use an alternative approach to derive the same relationships.

The 2D Fourier transform $M_k(v_m)$ of the modified sinogram at frequency $v_m$ can be calculated as

$$M_k(v_m) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} M(v_m, \phi) \exp(-jk\phi) d\phi. \quad (B1)$$

Substitution of Eqs. (A9a) and (A8a) of Appendix A into (B1) yields

$$M_k(v_m) = \exp[-jk \arcsin(v_a/2v_0)] \times \frac{1}{2\pi} \int_{\phi_0=0}^{2\pi} P(v_a, \phi_0) \exp(-jk\phi_0) d\phi_0$$

$$= \exp[-jk \arcsin(v_a/2v_0)] P_k(v_a). \quad (B2)$$

On the one hand, we have

$$\exp[-j \arcsin(v_a/2v_0)]$$

\begin{align*}
&= \cos \left[ \arcsin \left( \frac{v_a}{2v_0} \right) \right] - j \sin \left[ \arcsin \left( \frac{v_a}{2v_0} \right) \right] \\
&= \left( 1 - \frac{v_a^2}{4v_0^2} \right)^{1/2} - j \frac{v_a}{2v_0}. \quad (B3)\end{align*}

On the other hand, using the expressions for $\gamma(v_m)$ [Eq. (20)] and $v_\mu$ [Eq. (16)], we obtain

$$[\gamma(v_m)]^{-1} = \frac{v_m}{[v_m^2 + (\sqrt{v_0^2 - v_m^2} - v_0)^2]^{1/2}}$$

$$- j \frac{v_0 - (v_0^2 - v_m^2)^{1/2}}{[v_m^2 + (\sqrt{v_0^2 - v_m^2} - v_0)^2]^{1/2}}. \quad (B4)$$

Substitution of $v_m = v_a(1 - v_a^2/4v_0^2)^{1/2}$ into Eq. (B4) yields

$$[\gamma(v_m)]^{-1} = \left( 1 - \frac{v_a^2}{4v_0^2} \right)^{1/2} - j \frac{v_a}{2v_0}. \quad (B5)$$

A comparison of Eqs. (B3) and (B5) indicates that

$$\gamma(v_m) = \exp[j \arcsin(v_a/2v_0)]$$

and hence that

$$P_k(v_m) = [\gamma(v_m)]^2 M_k(v_m), \quad (B6)$$

which is identical to Eq. (22).

Similarly, the 2D Fourier transform $M_k(v_m)$ of the modified sinogram at frequency $-v_m$ can be calculated as

$$M_k(-v_m) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} M(-v_m, \phi) \exp(-jk\phi) d\phi. \quad (B7)$$

Substitution of Eqs. (A8b) and (A9b) of Appendix A into (B7) yields
\[ M_k(-v_m) = \exp(-jk \pi) \exp \left( jk \arcsin \left( \frac{v_a}{2v_0} \right) \right) \times \frac{1}{2\pi} \int_{\phi_0}^{2\pi} P(v_a, \phi_0) \exp(-jk \phi_0) d\phi_0 \]

\[ = (-1)^k \exp(j \arcsin(v_a / 2v_0)) P_k(v_a). \]  

(B8)

Noticing \( \gamma(v_m) = \exp[j \arcsin(v_a / 2v_0)] \) in Eq. (B8) yields

\[ P_k(v_a) = (-1)^k \left[ \gamma(v_m) \right]^{-k} M_k(-v_m), \quad (B9) \]

which is identical to Eq. (23).

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