RECOVERY OF COMPLEX IMAGES FROM FOURIER MAGNITUDE

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A computational investigation of the recovery of a complex image (i.e. an image whose pixels can have arbitrary phases) from the modulus of its Fourier transform is presented. The results indicate that an acceptable level of recovery can be obtained using Fienup's iterative algorithms, provided a good, but not necessarily exact, estimate of the image support (which does not need to be of a special form) is obtainable. The problem of poor correlation between subjective restoration quality and numerically calculated error measures is addressed, suggesting a more effective combination of established iterative algorithms. Results are given for several specimen images.

The problem of retrieving an object from the modulus of its Fourier transform arises in a number of seemingly disparate areas. Astronomy, electron microscopy, crystallography and wavefront sensing are all practical situations which would benefit from improved techniques for recovering the Fourier phase from the Fourier modulus.

There must, of course, be other a priori constraints on the image apart from knowledge of its Fourier magnitude or else arbitrary phase distributions could be chosen to produce an infinite variety of possible images. Fortunately, with the major exception of crystallography, most objects are of compact support, i.e. they are only non-zero within a finite region of space. The joint constraints of Fourier magnitude and image support provide the most general instance of what has been called the "Fourier phase problem" [1]. In this problem it is assumed that the Fourier magnitude can be sampled sufficiently finely to permit the autocorrelation of the object to be computed immediately, thereby giving an accurate estimate of the object's support [2].

A number of direct algorithms for phase recovery have recently been suggested [3-5], and these can all be applied to the general case of recovering a complex image (i.e. an image whose pixels can have arbitrary phases) from the modulus of its Fourier transform. It has, however, been noted that direct methods, relying as they do on the factorization of multi-dimensional polynomials, are susceptible to noise [6]. Although it appears hopeful that this difficulty may eventually be overcome [7], there is little doubt that iterative methods may often prove more robust.

It should be emphasised that the algorithms used in this paper were all developed by Fienup [8,9] and currently constitute the most practical technique for phase retrieval. Until now these algorithms have been thought to be ineffective unless the image is also known to be positive (i.e. real and non-negative) or have a special support [9]. This paper discusses how these existing algorithms can be used to recover complex images.

Consider two two-dimensional spaces called image-space and Fourier-space respectively, in which arbitrary points are identified by cartesian coordinates \((x, y)\) and \((u, v)\) respectively. Quantities existing in image-space are called images and are represented by lower case letters, e.g. \(f(x, y)\). The Fourier transform of an image exists in Fourier-space and is represented by the corresponding upper-case letter e.g. \(F(u, v)\). Thus

\[
f(x, y) \leftrightarrow F(u, v) = \iint f(x, y) \exp[-i2\pi(ux + vy)] \, dx \, dy,
\]

(1)
where $\leftrightarrow$ interconnects the members of a two-dimensional Fourier transform pair. $S_f(x, y)$ is the support of $f(x, y)$, i.e.

$$S_f(x, y) = \begin{cases} 1 & \text{where } |f(x, y)| > 0, \\ 0 & \text{where } |f(x, y)| = 0. \end{cases}$$ (2)

In general one has to make an estimate $B(x, y)$ of the support. Ideally this should exactly equal $S_f(x, y)$ but in general the estimated support will be larger than the true support $[2]$.

$$S_f(x, y) \in B(x, y).$$ (3)

It is worth noting that the image cannot be recovered exactly, but only what has been called the image-form $[1]$. If $(x_1, y_1)$ and $(x_2, y_2)$ are fixed points and $\omega_1$ and $\omega_2$ are arbitrary real constants, $f(x, y) = f(x-x_1, y-y_1) \exp(i\omega_1)$ and $f^*(x-x_2, y-y_2) \exp(i\omega_2)$, where the asterisk denotes complex conjugation, are all said to possess the same image-form. The name arises as the appearance (or form) of an image is unaltered by changing its location in image-space, and/or by rotating it 180 degrees in the coordinate origin and conjugating its phase and/or by adding a constant to its phase.

This paper uses the Fienup algorithms which have been well covered in previous work $[8,9]$ but are summarised below for completeness. Fig. 1 shows the iterative loop used in the Fienup algorithms and provides a summary of the notation used to describe the algorithms in this paper.

The convergence of an algorithm can be monitored in either image-space or Fourier-space. The normalised image-space metric is the most convenient and is given by $[8]$.

$$E_i = \left( \int_{S_f(x,y)} |g'(x, y)|^2 \, dx \, dy \right) \cdot \left( \int_{(u,v)} |g'(x, y)|^2 \, dx \, dy \right)^{-1}.$$ (4)

The two major algorithms differ in the formation of the new input $g_{k+1}(x, y)$. The error reduction algorithm forms a new input using

$$g_{k+1} = g_k(x, y), \quad (x, y) \in S_f(x, y),$$

$$= 0, \quad (x, y) \notin S_f(x, y).$$ (5)

This choice of the new input can be theoretically guaranteed not to increase the image-space error $[8]$. It can however stagnate, and in practice this has required the error reduction algorithm to be used in combination with other forms of input selection. The most effective of these is the hybrid input-output algorithm which starts a new iteration with

$$g_{k+1} = g_k(x, y), \quad (x, y) \in S_f(x, y),$$

$$= -g_k(x, y) - \beta g_k(x, y), \quad (x, y) \notin S_f(x, y).$$ (6)

where $\beta$ is a constant feedback parameter $[8]$. This does not have the strong theoretical basis of error-reduction but there appears little doubt that it is more effective in practice. The objective of this algorithm is to produce an input whose Fourier transform's phase is the same as that of the true image. It should be noted, however, that $G_k(u, v)$, although having the correct phase, can have a totally incorrect modulus, even when the algorithm has converged.

It has been observed that the correlation between the image quality and the error metric can be poor when the hybrid input–output algorithm is employed $[8]$. A well known example is the choice of a constant zero-phase as a starting estimate of the unknown phase. This yields a low starting error but results in a poor reconstruction $[8]$. In the past a combination of hybrid input–output and error-reduction has been employed to give a uniformly decreasing error curve, it is the experience of the author that the combination of error-reduction and hybrid input–output can be less effective than when hybrid input–output alone is employed.
It has been previously thought to be impracticable to recover a complex image-form from the Fourier modulus for centrosymmetric supports \([9,10]\), such as the circular image shown in fig. 2 (the phase of the image was randomly distributed between \(-\pi\) and \(+\pi\) and hence has not been shown). The image was recovered exactly (i.e. to an accuracy bounded by numerical precision used) from its Fourier modulus and support within 2000 iterations of hybrid input-output with \(\beta = 0.3\) (this is hereafter be referred to as the "hybrid only" approach). The dominant features of the image were visible after 1000 iterations. For comparison, a more traditional approach of error-reduction combined with hybrid input-output was employed (hereafter referred to as the "mixed" approach). This "mixed" approach consisted of an initial 20 iterations of error reduction followed by \(K\) cycles of hybrid input-output and error reduction, where a cycle consists of 40 iterations of hybrid input-output with \(\beta = 0.7\), followed by 10 iterations of the error reduction algorithm. Fig.

![Fig. 3. Image space error (eq. (4)) curves for reconstruction of image shown in fig. 2, using exact support. "mixed" approach, — "hybrid only" approach.](image)

Perhaps more significantly, both approaches recovered the \(32\times32\) pixel image shown in fig. 4 when \(B(x, y)\) was enlarged to \(33\times33\) pixels. No convergence was apparent within 10000 iterations using the "mixed" approach and a \(34\times34\) pixel support or contrast the "mixed" approach showed a sharp initial reduction in the image-space error whenever the error-reduction algorithm was applied but this was negated after further application of hybrid input-output (although not shown in fig. 3 the "mixed" approach was continued for \(K=200\) cycles, equivalent to 10000 iterations, with no apparent convergence).

![Fig. 4. Magnitude, quantised as for fig. 2, of a \(32\times32\) pixel complex image.](image)
the "hybrid only" approach and a 35×35 support. The comparison of the image-space errors for the "hybrid only" approach and the different support constraints is shown in fig. 5 and compares favourably with the "mixed" approach shown in fig. 6.

As a final comparison a positive image with a triangular support was chosen, fig. 7. The triangular support is known to help convergence, and the recovery of a complex image-form with this particular support has been previously demonstrated [9]. Positivity was not enforced and three error measures were compared from six different random starting phases.

Firstly the conventional image-space error (eq. (4)) was taken and the range from the minimum to the maximum error over the ensemble of six runs for the "hybrid only" and a "mixed" approach were plotted at intervals of 10 iterations, fig. 8. The "mixed" approach cycle was changed to 20 iterations of hybrid input–output with $\beta=0.7$, followed by 10 of error reduction. The image-space error curve for the "mixed" approach shows a characteristic dip upon application of error reduction after a period of hybrid input–output with a subsequent increase after reaplication of hybrid input–output. The "hybrid only" approach, by contrast, decreased steadily, although this would not be expected with a less restrictive support. Using this conventional error measure the "hybrid only" approach appears marginally better, but not significantly so.

Secondly, a direct RMS error between the true image and the reconstruction was taken.
Fig. 9. Direct image error (eq. (7)) plotted as described in text for reconstruction of image shown in fig. 6. ● ● "mixed approach, △-△ "hybrid only approach".

\[ E_d = \left( \int_{S(x,y)} \left| (g'(x,y) - f(x,y)) \right|^2 \, dx \, dy \right) \]

\[ \times \left( \int_{(x,y)} \left| f(x,y) \right|^2 \, dx \, dy \right)^{-1}. \]  

(7)

Although this measure is only applicable in simulations, it provides a close match to the visual quality of the images and provides a fairer means of comparing algorithms. Fig. 9 shows this direct error measure plotted at 10 iteration intervals in the same manner as fig. 8. It can be seen that the "hybrid only" approach is now significantly better. Of interest is that the jumps in the image-space error metric (eq. (4)) after reapplication of hybrid input-output after a period of error reduction are not mirrored in this direct error metric (eq. (7)), implying that hybrid input-output can achieve improvements in image quality which are not readily apparent in the conventional image-space error metric (eq. (4)).

Finally each run was assessed visually at 10 iteration steps by a different observer. In all cases the image reconstructed from the "hybrid only" approach was chosen to be closer to the original at the same iteration. The reconstruction was described as "good" at 40 iterations of the "hybrid only" approach and was comparable with 60 iterations of the "mixed" approach.

Conclusions. It has been shown that complex image forms can be recovered from their Fourier modulus even when the support estimate is inexact. The method proposed is simply a different combination of techniques developed by Fienup for the recovery of positive images, and is shown to be superior when compared visually and numerically to the true image, on the examples that have been tried.

This paper has not attempted to report the effects of noise on the reconstruction performance, and thus further research will be directed towards evaluating how the signal-to-noise affects the relative performance of the algorithms.

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