Toward reconstructing phases of inverse-scattering signals

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Recent work on the Fourier-phase problem is shown to be relevant to the reconstruction of the relative phases of fields scattered to a large number of separated inaccurately surveyed locations. A proposed approach to phase retrieval is illustrated with a computational example. It is assumed that the waveforms of the scattered fields can be recorded at each location.

1. INTRODUCTION

Most of the algorithms1 that have been developed for inferring the characteristics of bodies from scattered fields require the latter's phases to be known as well as their intensities. However, it is far from easy to record accurately the relative phases of electromagnetic signals received at widely separated locations. It is consequently worth inquiring whether such phases can be computed from measured intensities. We show here how recent results pertaining to the Fourier-phase problem2 might be adapted to the recovery of inverse-scattering phases.

Pertinent features of the Fourier phase problem are summarized in Section 2. Certain technical aspects of inverse scattering are discussed in Section 3, and in Section 4 we formulate the problem of inferring the relative phases of scattered fields received at a number of separated points lying in a plane. Section 5 presents a computational example that serves both to explain our proposed approach to phase reconstruction and to demonstrate that it shows promise of being useful in practice. We assess the significance of our results in Section 6 and outline the kinds of investigation that are needed in the future.

2. CONNECTION BETWEEN FOURIER PHASE AND INTENSITY

There are two virtually distinct phase problems associated with pairs of quantities that are each other's Fourier transforms.6 The older of these, known as the crystallographic-phase problem,8 cannot be attacked by the techniques summarized in this section, for a reason (both illuminating and important) that is explained later. Recent research indicates that the second problem, which we call the Fourier-phase problem, is appositely formulated as follows. First, we are presented with a positive (i.e., real and nonnegative) quantity $I(u)$ that is well behaved throughout an $L$-dimensional space (here called Fourier space), and its Fourier complex amplitude, and $|F(u)|$ the Fourier magnitude. We next demand that the support of the image be effectively finite, in the sense that if

$$|f(x)| < \epsilon \quad \text{for all } x \in T$$

(2)

where $\epsilon$ is some positive constant and $T$ is a region (here called the image box) of image space, then the integral (here called the $L$-dimensional image volume $V$) over image space of the quantity, which is unity for $x \in T$ and zero for $x \notin T$, is finite. Finally, we require $V$ to be the smallest image volume compatible with Eq. (1).

It is a mathematical theorem that

$$I(u) \leftrightarrow f(x),$$

(3)

where $f(x)$ is the autocorrelation of $f(x)$. It is appropriate to introduce the autocorrelation box $T_\sigma$, defined by

$$|f(f(x))| < \eta \quad \text{for all } x \in T_\sigma,$$  

(4)

where $\eta$ is another positive constant. The autocorrelation volume $V_\sigma$ is the integral over image space of the quantity that is unity for $x \in T_\sigma$ and zero for $x \notin T_\sigma$.

There is no simple connection between $T$ and $T_\sigma$, in general, when $\epsilon$ and $\eta$ are chosen arbitrarily. However, when the following conditions hold (which we believe they must in the majority of physical applications of the concepts being outlined here), there is a particularly simple connection. Suppose that the form of $I(u)$ is such that, first, $V$ and $V_\sigma$ can both remain finite, with $\epsilon$ and $\eta$ being as small as we please, and, second, the surfaces $\sigma$ and $\sigma_\eta$ bounding $T$ and $T_\sigma$, respectively, are both single connected and convex. Now consider the $L$-dimensional rectangular parallelepipeds $\Lambda$ and $\Lambda_\sigma$ that just enclose $T$ and $T_\sigma$, respectively. The sides of $\Lambda$ and $\Lambda_\sigma$ are parallel to Cartesian axes set up in image space. We denote by $b_\lambda$ and $a_\sigma$, respectively, the lengths of the sides of $\Lambda$ and $\Lambda_\sigma$ in the $l$th direction. It is obvious from the form of the integral that defines the autocorrelation that

$$[I(u)]^{1/2} \exp[i\Phi(u)] = F(u) \leftrightarrow f(x),$$

(1)

where $L$ connects the members of an $L$-dimensional Fourier-transform pair, $x$ is the position vector of an arbitrary point in a second $L$-dimensional space (here called image space), and $i = \sqrt{-1}$. It is appropriate to call $f(x)$ the image, $F(u)$ the Fourier complex amplitude, and $|F(u)| = [I(u)]^{1/2}$ the Fourier magnitude. We require $V$ to be the smallest image volume compatible with Eq. (1).
It is appropriate to set up Cartesian axes in Fourier space also, with the correspondence between the $l$th coordinate $u_l$ and the $l$th coordinate $x_l$ in image space being defined by the Fourier integral formula

$$F(u) = \int \left( \frac{L}{i} \right) \int f(x) \exp(i2\pi u \cdot x) d\gamma$$

where $\int (L) \int$ implies an $L$-dimensional integral and $d\gamma$ is the volume element in image space.

In practical applications, few physical quantities are recorded as continuous functions of time and/or distance. They are usually sampled in both time and space. The theorem represented by expression (3) thus can be taken advantage of in the real world only if the points (here called the sample points) belonging to the set $|u_j; j = 1, 2, \ldots, J|$ at which $f(u)$ is given (or measured) are spaced sufficiently closely. The sampling theorem demands that the spacing in the $l$th Cartesian direction be no greater than $1/\alpha_l$. It is this that distinguishes the crystallographic and Fourier phase problems. For the former, the corresponding sample spacing is unavoidably larger, because crystallographic images are (by definition) periodic, repeated throughout image space within contiguous parallelepipeds (which are, in fact, nonrectangular, in general). When these parallelepipeds are rectangular, the Fourier intensity is observable only at points spaced by $1/\alpha_l$ in the $l$th direction. Equation (5) shows this spacing to be twice $1/\alpha_l$.

Given only the Fourier intensity, it is impossible to find where the image is located. Furthermore, one cannot distinguish between the image and the complex conjugate of its reflection in the coordinate origin. However, an object is not changed by shifting it, and its reflection remains recognizable even when its phase is reversed. We say that $f(x-y)$ and $f^*(x+z)$ have the same image form as $f(x)$, where $y$ and $z$ are arbitrary constant position vectors and the asterisk denotes complex conjugation.

When some Fourier phase information is available, as in the situation envisaged in Section 4, we can choose between $f(x)$ and $f^*(-x)$. The uncertainty over the image's position persists, however, but this has no significance in inverse-scattering contexts.

When $L = 1$, there is no unique connection between the Fourier intensity and the image form. When $L > 1$, however, there is almost always a unique connection (in fact, the available evidence strongly suggests that, in practice, the connection is always unique for the image form whose support is as compact, as is consistent with the data).

3. INVERSE SCATTERING

Consider a finite body, say, $B$ (refer to Fig. 1), illuminated with prescribed electromagnetic radiation. We denote a particular linearly polarized component of the scattered field (received at the point $P$) by the signal $s(r, \phi, \theta, t)$, where $t$ denotes time. If, as is usual, $P$ is in the far field (Fraunhofer region) of $B$ for all significant frequencies present in the signal, then the $r$ dependence becomes trivial, and the non-trivial functional dependence of the signal is expressed by $s(\phi, \theta, t)$, the temporal Fourier transform of which is written as $\tilde{s}(\phi, \theta, k)$, where $k = 2\pi/\lambda$ is the wave number corresponding to the free-space wavelength $\lambda$. So,

$$\tilde{s}(\phi, \theta, k) = s(\phi, \theta, t).$$

The scattered field can be rigorously regarded as reradiations from equivalent sources, appropriately called polarization sources, initially induced in the body by the incident illumination. The reradiations induce further sources, etc., so that the actual equivalent-source distribution is self-consistent with both the illumination and the constitution of the body.

When, as is usual in the real world, it is feasible to sense the scattered field only within restricted ranges (of extent appreciably less than $\pi$) of the angles $\phi$ and $\theta$ (we call these the angular measurement ranges), we can introduce a theoretical simplification that, although approximate, nevertheless often gives accurate results in practice. A fictitious plane (here called the equivalent-image plane) is set up contiguous to the aforesaid polarization source distribution, with the latter on the side opposite that of the sensors of the scattered field. If the correct forms are adopted for equivalent surface sources on the whole (infinite extent) plane, the far field radiated by them exactly equals $s(\phi, \theta, t)$. This is one way of expressing Huygens's principle mathematically (cf. Baker and Copson).

The above-mentioned exact formulation is of little help, however, when the sensors cover only a restricted field of view. One must then resort to the following device, which is actually implicit in virtually all useful interpretations of the images formed by microscopes, theodolites, telescopes, eyes,
identifies the part of the equivalent-source distribution (we
call this the support of the equivalent image) that is
reconstructable from measurements made with the sensors. The recon-
struction cannot be exact, of course, but it is quite often
palpably faithful. This principle is implicit, for example, in
the approximate physical-optics formulation of inverse scatter-
ing and in the conceptual foundations of successful micro-
wave-imaging techniques. Our ultrasonic-imaging
experiments are explicit demonstrations of the effective-
ness of this concept. One must, of course, recognize that the
size of any detail that can be resolved is necessarily inversely
proportional to the extents of the angular measurement
ranges.

The Fourier phase problem (see Section 2) is formulated
in terms of images having compact support. Since signals
tend to be of palpably finite duration, it is appropriate to
treat $t$ as one image coordinate. Because the support of the
equivalent image is also effectively finite, we take Cartesians
($x$ and $y$, say) in the equivalent-image plane to be the other
two image coordinates. To ensure that the equivalent image
$s(x, y, t)$ is the Fourier transform of the scattered field, the
Fourier-space coordinates must be $k$ and $u = [\sin(\theta)\times \cos(\phi)]/\lambda$ and $v = [\sin(\theta)\sin(\phi)]/\lambda$, where the pole of the
spherical system ($r, \phi, \theta$) is to be thought of as passing
through a central point of the angular measurement ranges.
So $x$, $y$, and $t$ and $u$, $v$, and $k$ are the respective components
of the three-dimensional vectors $x$ and $u$. We write $s(\phi, \theta, k)$ as
$S(u)$.

We envisage that signals are received at points spaced in
the $u$ and $v$ directions, respectively, by no more than one half
of the reciprocal of the linear extent in the $x$ and $y$ directions
of the equivalent image. This is necessary to permit the
phase-retrieval techniques outlined in Section 2 to be ap-
plied.

Since the measurement ranges are finite, the observable
$f(x, y, t)$ is truncated. This does not prevent imaging from
being performed in practice, as explained above. It is advis-
able, however, to ameliorate the effects of the truncation by
appropriate preprocessing, such as windowing or edge exten-
sion of the observed data$^{12}$ (the latter technique is often
quite efficacious$^{12}$). We denote the preprocessed form of
$S(u)$ by

$$F(u) = W[S(u)],$$

where $|W[\cdot]|$ represents whatever form of preprocessing is
adopted.

An exceedingly pertinent practical point is that $f(x)$ is
taken to be positive (i.e., real and nonnegative) in all reports
of successful phase retrieval.$^2$ In general, there is no reason
to expect that a $F(u)$ pertaining to inverse-scattering data is
the Fourier transform of a positive quantity. This would
mean that reconstruction of inverse-scattering data would
be impractical if no part of the phase of $F(u)$ could be
measured directly. Happily, a considerable amount of phase information is immediately inferable from measure-
ment, as is explained in Section 4.

4. TWO-DIMENSIONAL CONSIDERATIONS

To ensure that $|F(u)|$ is uniquely connected with its image
form, the dimension of the position vector $u$ need be no more
than 2. From now on we assume that $P$ lies in the plane
defined by $\phi = 0$ and $\phi = \pi$, so that the signal's functional
dependence reduces to $s(\theta, t)$, and $u$ has the two components
$u$ and $k$. The respective image-space coordinates are $x$ and
$t$.

Although the waveform of the signal received at $P$ can be
readily recorded for frequencies as high as those of the infra-
red$^4$ (and optical heterodyning will presumably soon be-
come routine), there are likely to be appreciable errors in
practice, however, in estimates of $r$ and the time origin for
each point $P$. This means that, although the major func-
tional dependence of $\Phi(u) = \Phi(u, k)$ is readily deducible from
the measured data, we must expect significant errors in
the average value and the slope (in the $k$ direction) of the
Fourier phase for each value of $u$ at which data samples are
available. So the estimate $\hat{\Phi}(u, k)$ of the Fourier phase
obtainable directly from the measurement can be expected
to be of the form

$$\hat{\Phi}(u, k) = \Phi(u, k) + \alpha(u) + k\beta(u).$$

The unknown $\beta(u)$ can be removed easily. We merely
Fourier transform the data one-dimensionally (with respect
to $k$) for each value of $u$ and then shift the transformed
values such that they all span the same region of $t$ values.
Consequently, the only serious errors in the data are repres-
ented by $\alpha(u)$, whose correction is described in Section 5,
wherein we explain our approach to phase reconstruction
with the aid of a computational example. For convenience,
we take $t$ to fill all $A$, whose size is normalized such that

$$b_1 = b_2 = 1 \quad \text{so that } a_1 = a_2 = 2.$$ (10)

We pose the inverse-scattering phase problem as follows:
Given values of $I(m/2, n/2)$ and $\hat{\Phi}(m/2, n/2)$, for sufficient
integers $m$ and $n$ that $f(x, t)$ can be reconstructed (to within
whatever accuracy is specified) from the given samples of
$I(u) = I(u, k)$, reconstruct the Fourier phase $\Phi(u, k)$.

5. PHASE RECONSTRUCTION: AN EXAMPLE

The concern here is solely with the reconstruction algorithm.
The data are not taken from an actual inverse-scattering
problem. What needs to be demonstrated at this stage is
that the Fourier phase can be readily retrieved when the
image is complex. So a complex $f(x) = f(x, t)$, existing for $|x|
< 1/2$ and $|t| < 1/2$ in accordance with Eq. (10) has been
chosen arbitrarily. The data for the inverse-scattering
problem were generated from the computer Fourier trans-
form of the image.

The first step is to deduce the size of $A_\alpha$ from the given
samples of $I(u, k)$. We take this as accomplished, indicating
that $A_\alpha$ has sides of length $2$. As explained in Section 2,
this shows immediately that the sides of $A$ are of length $1$.

The next step is to remove the error $k\beta(u)$ in the manner
described in Section 4. Since there is no difficulty in doing
this, we just assume that it is accomplished, so that the given
phase can be expressed in the simplified form:

$$\hat{\Phi}(u, k) = \Phi(u, k) + \alpha(u).$$

(11)
\[ \tilde{f}(x) = \tilde{f}(x) \quad \text{for} \ x \in \Lambda \]
\[ = 0 \quad \text{for} \ x \in \tilde{\Lambda}_a, \]  
\[(13)\]

where \( \tilde{\Lambda}_a \) is the part of \( \Lambda_a \) not occupied by \( \Lambda_a \).

Writing the FFT of \( \tilde{f}(x) \) as \( \tilde{F}(u) = \tilde{F}(u, k) \), we denote the phase of the latter by \( \tilde{\Phi}(u, k) \).

We must now use \( \tilde{\Phi}(u, k) \) to estimate the error \( \alpha(u) \) in \( \Phi(u, k) \) as given by Eq. (11). For each value of \( u \), we calculate the average difference between \( \Phi(u, k) \) and \( \tilde{\Phi}(u, k) \), which we denote by

\[ \Delta(u) = \frac{1}{N(u)} \sum_m [\Phi(u, m/2) - \tilde{\Phi}(u, m/2)], \]  
\[(14)\]

where \( N(u) \) is the number of data samples for the particular value of \( u \) (i.e., it is the number of values of \( k \) for which data are given for each value of \( u \)). We take \( \Delta(u) \) to be our estimate of \( \alpha(u) \). We call \( [\Phi(u, k) - \Delta(u)] \) the corrected phase.

By replacing \( \Phi(u, k) \) in Eq. (12) by the corrected phase, we set up an iterative loop, which is continued until \( |\tilde{f}(x)| \) is less than some prescribed threshold throughout \( \tilde{\Lambda}_a \). We call this procedure the modified Fienup algorithm. The standard Fienup algorithm is obtained by replacing \( \Phi(u, k) \) in Eq. (12) by \( \tilde{\Phi}(u, k) \).

Figure 2(a) shows \( |\tilde{f}(x)| \) for a complex image consisting of five “Gaussian blobs” whose relative phases were derived.

We now define \( \hat{F}(u, k) \) and \( \hat{f}(x, t) \) by

\[ \hat{F}(u, k) = |\hat{F}(u, k)| \exp(i\Phi(u, k)) \exp(\hat{f}(x, t)) = \tilde{\hat{f}}(x), \]  
\[(12)\]

where \( |\hat{F}(u, k)| \) is the positive square root of the given Fourier intensity. Since \( \tilde{\hat{f}}(x) \) is computed, using the fast-Fourier-transform (FFT) algorithm, from the given samples of \( F(u, k) \) and \( \Phi(u, k) \), which are spaced by 1/2 in both the \( u \) and \( k \) directions, this estimate of the image has value (in general) throughout \( \tilde{\Lambda}_a \). It is appropriate to define

\[ E(\text{degrees}) \quad \text{versus} \quad N \quad \text{of iterations}: \]

--- modified Fienup algorithm; --- standard Fienup algorithm.
from a pseudorandom distribution uniformly distributed between 0 and 2π. Figure 2(b) shows the magnitude of the first version of \( \hat{f}(x) \), which bears little resemblance to the original \( f(x) \), as is, of course, to be expected. After only four iterations, however, the form of \( \hat{f}(x) \) is similar to that of \( f(x) \), as is shown in Fig. 2(c). Figure 3 shows how the mean phase error \( \bar{E} \) varies with the number \( N \) of iterations, where \( E \) is the sum over all pixels in Fourier space of \( |\Phi(u, k) - \hat{\Phi}(u, k)| \) divided by the number of pixels.

The modified Fienup algorithm converges rapidly, as is seen from Fig. 3. In fact, the tenth version of \( |\hat{f}(x)| \) is virtually indistinguishable from \( |f(x)| \). On the other hand, the standard Fienup algorithm does not seem to converge at all, as is confirmed in Fig. 3. For instance, at the tenth iteration the reconstructed image form is much the same as that shown in Fig. 2(b). This does not indicate any inadequacy on the part of the standard algorithm, of course, because \( f(x) \) is complex here.

6. CONCLUSIONS

The first significant point worth noting is that the dashed curve in Fig. 3 confirms the general (although largely unrecorded, as far as the archival literature is concerned) experience of those working on Fourier-phase problems that recovery of complex image forms from their Fourier intensities is difficult. Complex images reconstructed by standard Fienup algorithms always seem to spread themselves more or less uniformly throughout \( \Lambda \). This shows just how powerful is the constraint of “positivity,” which is invoked in all previous reports of successful phase retrieval.²

The encouraging conclusion to be drawn from this paper is that, even when \( f(x) \) is complex, the Fourier phase can be retrieved accurately, provided that the available estimate of it can be expressed as in Eq. (9). This suggests that it may well prove to be feasible to reconstruct the relative phases of scattered signals received at widely separated sites, thereby significantly enhancing the physical relevance of most of the existing inverse scattering algorithms.¹

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REFERENCES


7. S. Silver, Microwave Antenna Theory and Design (Dover, New York, 1965), Chap. 5.


10. B. D. Steinberg, Microwave Imaging with Large Antenna Arrays (Wiley, New York, 1983).


(see overleaf)
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