COMPARISON OF MODULATION OF PERIODIC TARGETS IN PARTIALLY COHERENT LIGHT

by

P. K. MONDAL and V. VENKATESWARA RAO

and

Serge SLANSKY

INTRODUCTION

The advantage of using a sine-wave input for transfer function measurements is that it gives a sinusoidal brightness variation in the image, and theoretically that is the simplest object to study. Experimentally however, there are considerable difficulties in obtaining a test target with unit contrast and sinusoidal transmission over a wide frequency range [1-2]. As a consequence, attention was turned towards making use of targets with square-wave transmission [3]. Here also one encounters with another difficulty from the theoretical point of view. The Fourier series representation of a square-wave grating is not free from the unwanted Gibbs phenomenon [4]. A complete and up-to-date review of the use of sine-wave and square-wave targets for OTF measurements by various workers can be found in a few papers by Katti, Singh and others [5-6]. In order to remove the difficulties encountered with the sine-wave and square-wave targets, Lohmann has proposed the use of triangular-wave targets [7-8], since this is almost free from the unwanted Gibbs phenomenon and can also be realised experimentally by sharpening moiré fringes [9]. Françon and his co-workers [10] have recently proposed a simple photographic method of producing triangular-wave targets. Considerable interest has, therefore, been shown by various workers in using triangular-wave targets and a detailed account of this can be found in the reference [5]. Studies on incoherent response of these targets by slit, annular and rectangular apertures with various amplitude filters and aberrations have been made by Katti, Singh and others [11-13].

In view of what has been said above, it is desirable to compare the modulations of the three types of targets under different conditions of illumination. In this paper, we have presented the results of our theoretical investigations on the modulations of sine-wave, square-wave and triangular-wave targets in partially coherent light.
THEORY

The analysis is based on the theory of image formation in optical instruments in partially coherent light as formulated by Hopkins [14]. The optical system can be represented in a schematic manner as shown in the figure 1. The actual source and the condenser are replaced by an effective source having the intensity distribution \( y(x_0, y_0) \) in the exit pupil of the condenser. The object is defined by its complex amplitude transmission \( A(u, U) \) and the objective forms an image with the intensity distribution \( B'(u', U') \). The reduced coordinates \( (U, U) \) in the object plane are defined by

\[
U = \left( \frac{2 \pi}{\lambda} \right) n \sin \alpha \xi \quad \text{and} \quad U = \left( \frac{2 \pi}{\lambda} \right) n \sin \alpha \eta
\]

where \((\xi, \eta)\) are the cartesian co-ordinates in the object plane, \(\lambda\) and \(n\) sin \(\alpha\) have their usual significances. \((u', v')\) represent the corresponding co-ordinates in the image plane. The coordinates \((x, y)\) on the pupil of the objective are normalised in such a manner that the effective source is a circle of unit radius. The coordinates \((x_0, y_0)\) in the pupil of the condenser are such that the radius \(\rho\) of the effective source is equal to the ratio of the numerical aperture of the condenser to that of the objective, i.e.

\[
\rho = n \sin \alpha \xi / (n_0 \sin \alpha_0).
\]

The degree of coherence on the object plane will be governed by the radius \(\rho\) of the effective source. The two limiting cases \(\rho \to 0\) and \(\rho \to \infty\) correspond to coherent and incoherent illumination, respectively.

The distribution of intensity in the image of the trans-illuminated object is given by:

\[
B'(u', v') = \int \int_{\Sigma} y(x_0, y_0) \left| A'(x_0, y_0; u', v') \right|^2 \, dx_0 \, dy_0.
\]

The expression (6) is very convenient for computing diffraction images of non-periodic objects. An alternative form suitable for periodic objects also has been shown by Hopkins [14]. Let a periodic object be defined by the complex amplitude transmission

\[
A(u, v) = \sum_n a_n e^{i m u}
\]

the radial spatial frequency \(\omega\) being related to the period \(p\) by \(\omega = 2 \pi / p\).

The intensity distribution in the image is given by

\[
B'(u', v') = \sum_m C(m, n) a_m a^*_n \sum_n e^{i(m-n)\omega u'}
\]

where \(\omega\) denotes the complex conjugate. \(C(m, n)\) is defined by

\[
C(m, n) = \int \int_{\Sigma} y(x_0, y_0) \left| f(x_0 + m \omega, y_0) \right|^2 \, dx_0 \, dy_0.
\]

The integration having been taken over the area \(\Sigma\) of the effective source. Here \(\gamma(x_0, y_0)\) is the intensity distribution over the effective source, whilst \(A'(x_0, y_0; u', v')\) represents the complex amplitude at the point \((u', v')\) of the image plane due to a point \((x_0, y_0)\) of the effective source, and is given by

\[
A'(x_0, y_0; u', v') = \int \int_{pupil} a(x - x_0, y - y_0) f(x, y) e^{i(m u + n v)} \, dx \, dy \quad \text{in which} \quad f(x, y) \quad \text{denotes the pupil function, and} \quad a(x, y)
\]

is the Fourier transform of the complex amplitude transmission of the object structure.

If we assume the exit pupil of the condenser to be uniformly illuminated, we can take \(\gamma(x_0, y_0) = 1\), and the equation (3) reduces to

\[
B'(u', v') = \sum_m \left| A'(x_0, y_0, u', v') \right|^2 \, dx_0 \, dy_0.
\]

PERIODIC TARGETS

\[
B'(u', v') = \sum_m \sum_n \left| A'(x_0, y_0, u', v') \right|^2 \, dx_0 \, dy_0.
\]

where \(m\) denotes the complex conjugate. \(C(m, n)\) is defined by

\[
C(m, n) = \int \int_{\Sigma} f(x_0 + m \omega, y_0) \times f^*(x_0 + n \omega, y_0) \, dx_0 \, dy_0
\]

and is known as the generalised frequency response of the optical system in partially coherent light. By grouping the terms corresponding to the same frequency in the image, one can write

\[
B'(u', v') = \sum_n b_n e^{i m u}
\]

SOLUTION FOR PERIODIC TARGETS
the coefficients $b_n$ being defined by

$$b_n = \sum_m C(m, m - n) a_m a^*_m a_{m-n}.$$ 

(a) Sine-wave targets

The complex amplitude transmission of the object is

$$A(u) = (1/2)(1 + \cos \omega u) = a_0 + a_1 e^{iu} + a_{-1} e^{-iu}.$$ 

There are only three frequencies, $0$, $\omega$, and $-\omega$, with the coefficients

$$a_0 = 1/2, \quad a_1 = a_{-1} = 1/4.$$ 

The distribution of intensity in the image is

$$B'(u') = |a_0|^2 C(0, 0) +$$

$$+ |a_1|^2 C(1, 1) + |a_{-1}|^2 C(-1, -1) +$$

$$+ [a_1 a^*_0 C(0, 1) + a_0 a^*_1 C(0, -1)] e^{iu'}$$

$$+ [a_0 a^*_0 C(1, 0) + a_1 a^*_0 C(-1, 0)] e^{-iu'} +$$

$$+ a_1 a^*_1 C(1, -1) e^{-2iu'} + a_{-1} a^*_1 C(-1, 1) e^{2iu'}.$$ 

This expression can be simplified by taking account of the following considerations. According to the definition (9), $C(m, n)$ and $C(n, m)$ are complex conjugates

$$C(m, n) = C^*(n, m).$$ 

Furthermore, in a stigmatic system the function $f(x, y)$ is uniform over the pupil, so that we can write

$$C(-m, -n) = C(m, n).$$ 

Taking account that $a_0$ and $a_1 = a_{-1}$ are real, the expression (13) simplifies to

$$B'(u') = a_0^2 C(0, 0) + 2a_1^2 C(1, 1) +$$

$$+ 4a_0 a_1 C(0, 1) \cos \omega u' +$$

$$+ 2a_1^2 C(1, -1) \cos (2 \omega u').$$

Introducing the values of $a_0$ and $a_1$, we obtain

$$B'(u') = (1/8)[2C(0, 0) + C(1, 1) +$$

$$+ 2C(0, 1) \cos \omega u' + C(1, -1) \cos (2 \omega u')]$$

(b) Square-wave targets

The complex amplitude transmission $A(u, v) = A(u)$ within a period $(-p/2, p/2)$ is given by

$$A(u) = 1 \quad \text{for} \quad |u| \leq p/4$$

$$= 0 \quad \text{for} \quad p/4 < |u| \leq p/2$$

and $A(u + p) = A(u)$.

This function can be represented by the series

$$A(u) = 1/2 \times \frac{(\cos \omega u - (1/3) \cos (3 \omega u) + ...)}{\cos \omega u}.$$ 

The distribution of intensity in the image can be obtained by introducing in the general expression of $B'(u', v')$ the following values of the coefficients :

$$a_0 = 1/2, \quad a_1 = a_{-1} = 1/4.$$

$$a_{2m} = 0, \quad a_{2m+1} = (-1)^m/(2m + 1) \pi.$$ 

c) Triangular-wave targets

The function $A(u, v)$ within a period $(-p/2, p/2)$ is given by

$$A(u, v) = 1 - |2u/p|$$

the Fourier series representation of which is

$$A(u, v) = 1/2 + (2\pi)^2 \frac{\cos (\omega u) + \cos (3\omega u) + ...}{25 + ...}$$

$$= 1/2 + (2\pi)^2 \sum \cos ((2m + 1)\omega u)/(2m + 1)^2.$$ 

The coefficients $a_n = a_{-n}$ are therefore

$$a_0 = 1/2; \quad a_1 = 2\pi^2; \quad a_{2m} = 0; \quad a_{2m+1} = 2/[2m + 1 \pi]^2.$$ 

RESULTS

In the present study, we have supposed that the optical system forming the image is stigmatic with a circular pupil whose radius is equal to unity. We have, therefore

$$f(x, y) = 1 \quad \text{for} \quad x^2 + y^2 \leq 1$$

$$= 0 \quad \text{for} \quad x^2 + y^2 > 1.$$ 

The modulations in the images have been calculated according to the formula

$$\text{Modulation} = (B'_{\text{max}} - B'_{\text{min}})(B'_{\text{max}} + B'_{\text{min}})$$

where $B'_{\text{max}}$ and $B'_{\text{min}}$ are, respectively, the maximal and minimal values of the intensity in the image.

The final computations were made on a UNIVAC-1108 computer. For the sake of brevity, we avoid the details of the computations. We simply mention here that for given values of the frequency and the effective source radius one can calculate $C(m, n)$ according to the expression (9). In the case of a stigmatic optical system, $C(m, n)$ is proportional to the common area enclosed in three circles, one of radius $\rho$ centered at $(0, 0)$ and two others of radius unity centered at $(m\rho, 0)$ and $(n\rho, 0)$ respectively. Once $C(m, n)$ is known for different values of $m$ and $n$, the distribution of intensity in the images of periodic objects of different forms can be obtained by introducing in the expression of $B'(u', v')$ the values of the coefficients $a_n$ appearing in the series development of $A(u, v)$.

In figure 2, we have plotted the coherent responses of the three types of periodic targets against frequency. There is a sharp cut-off at $\omega = 1$ for all the targets. But, whereas the sine-wave and square-wave response is unity over the transmitted frequency-band, the triangular-wave response, as indicated by the broken line, is slightly less. Figures 3-7 show the responses of the square-wave ($- -$), sine-wave ($-----$) and triangular-wave ($- -$) targets in partially coherent light for various values of the radius $\rho$ of...
the effective source. The square-wave response is higher than the sine-wave and triangular-wave responses. In coherent or nearly coherent illumination, the differences between the responses of the three types of targets are not great. The differences increase when the radius of the effective source is increased, that is when the coherence is decreased. However, it has been shown by Katti, Singh and others [5] that a proper choice of the width of the bright portion of the target pattern can make the incoherent triangular-wave response approximately equal to the sine-wave response. They found that for the value of $\alpha = 0.45$ (where $\alpha p$ is the width of the bright portion at the mean irradiance and $p$ is the period of the pattern), the triangular-wave response can be used in the place of the sine-wave response. In our paper, we have considered equal widths of the bright and dark portions, which corresponds to $\alpha = 0.5$. 

FIG. 2. — Response curves of the periodic targets in coherent light.

Fig. 2. — Courbes de réponse des créneaux périodiques en lumière cohérente.

FIG. 3. — Square-wave (- - - - - -), sine-wave (-----) and triangular-wave (-----) response curves in partially coherent light for $p = 0.2$.

Fig. 3. — Courbes de réponse des mires en lumière partiellement cohérente pour $p = 0.2$ pour des surfaces rectangulaires et sinusoidales.

FIG. 4. — Square-wave (- - - - - -), sine-wave (-----) and triangular-wave (-----) response curves in partially coherent light for $p = 0.4$.

Fig. 4. — Courbes de réponse des mires en lumière partiellement cohérente pour $p = 0.4$ pour des surfaces rectangulaires et sinusoidales.

FIG. 5. — Square-wave (- - - - - -), sine-wave (-----) and triangular-wave (-----) response curves in partially coherent light for $p = 0.6$.

Fig. 5. — Courbes de réponse des mires en lumière partiellement cohérente pour $p = 0.6$ pour des surfaces rectangulaires et sinusoidales.

FIG. 6. — Square-wave (- - - - - -), sine-wave (-----) and triangular-wave (-----) response curves in partially coherent light for $p = 0.8$.

Fig. 6. — Courbes de réponse des mires en lumière partiellement cohérente pour $p = 0.8$ pour des surfaces rectangulaires et sinusoidales.

FIG. 7. — Square-wave (- - - - - -), sine-wave (-----) and triangular-wave (-----) response curves in partially coherent light for $p = 1.0$.

Fig. 7. — Courbes de réponse des mires en lumière partiellement cohérente pour $p = 1.0$ pour des surfaces rectangulaires et sinusoidales.
RÉFÉRENCES


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